

# Uniqueness of equivariant singular Bott-Chern classes

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**Abstract.** In this paper, we shall discuss possible theories of defining equivariant singular Bott-Chern classes and corresponding uniqueness property. By adding a natural axiomatic characterization to the usual ones of equivariant Bott-Chern secondary characteristic classes, we will see that the construction of Bismut's equivariant Bott-Chern singular currents provides a unique way to define a theory of equivariant singular Bott-Chern classes. This generalizes J. I. Burgos Gil and R. Lițcanu's discussion to the equivariant case. As a byproduct of this study, we shall prove a concentration formula which can be used to prove an arithmetic concentration theorem in Arakelov geometry.

**Résumé.** Dans cet article, nous allons discuter les théories éventuelles de définir les classes de Bott-Chern équivariantes singulières et la propriété d'unicité correspondante. En ajoutant une caractérisation axiomatique naturelle à lesquelles habituelles des classes caractéristiques secondaires de Bott-Chern équivariantes, nous verrons que la construction des courants de Bott-Chern équivariants singuliers de Bismut offre un moyen unique de définir une théorie des classes de Bott-Chern équivariantes singulières. Ceci généralise la discussion de J. I. Burgos Gil et R. Lițcanu dans le cas équivariant. En tant qu'un sous-produit de cette étude, nous allons prouver une formule de concentration qui peut être utilisée pour prouver un théorème de concentration arithmétique en géométrie d'Arakelov.

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## 1 Introduction

The Bott-Chern secondary characteristic classes were introduced by R. Bott and S. S. Chern. They can be used to solve the problem that the Chern-Weil theory is not additive for short exact sequence of hermitian vector bundles. More precisely, assume that we are given a short exact sequence

$$\bar{\varepsilon}: 0 \rightarrow \bar{E}' \rightarrow \bar{E} \rightarrow \bar{E}'' \rightarrow 0$$

of hermitian vector bundles on a compact complex manifold  $X$ . Then the alternating sum of Chern character forms  $\text{ch}(\bar{E}') - \text{ch}(\bar{E}) + \text{ch}(\bar{E}'')$  is not equal to 0 unless this sequence is orthogonally split. A Bott-Chern secondary characteristic class associated to  $\bar{\varepsilon}$  is an element  $\tilde{\text{ch}}(\bar{\varepsilon}) \in \tilde{A}(X)$  (cf. Section 2) satisfying

(i). (Differential equation)  $\text{dd}^c \tilde{\text{ch}}(\bar{\varepsilon}) = \text{ch}(\bar{E}') - \text{ch}(\bar{E}) + \text{ch}(\bar{E}'')$ . Here the symbol  $\text{dd}^c$  is the differential operator  $\frac{\bar{\partial}\partial}{2\pi i}$ .

J.-M. Bismut, H. Gillet and C. Soulé's construction of Bott-Chern secondary classes (cf. [1]) forces it to satisfy other two properties

(ii). (Functoriality)  $f^*\tilde{\text{ch}}(\bar{\varepsilon}) = \tilde{\text{ch}}(f^*\bar{\varepsilon})$  if  $f : X' \rightarrow X$  is a holomorphic map of complex manifolds.

(iii). (Normalization)  $\tilde{\text{ch}}(\bar{\varepsilon}) = 0$  if  $\bar{\varepsilon}$  is orthogonally split.

It has been shown that the three properties above are already enough to characterize a theory of Bott-Chern secondary characteristic classes. The same thing goes to the Chern-Weil theory in the equivariant case. We shall recall these results in Section 2 for the convenience of the reader.

In [3], J.-M. Bismut, H. Gillet and C. Soulé defined the Bott-Chern singular currents in order to solve a similar differential equation as in (i) with respect to the resolution of hermitian vector bundle associated to a closed immersion of complex manifolds. Later in [4], J.-M. Bismut generalized this topic to the equivariant case. Precisely speaking, let  $G$  be a compact Lie group and let  $i : Y \rightarrow X$  be a  $G$ -equivariant closed immersion of complex manifolds with hermitian normal bundle  $\bar{N}$ . Suppose that  $\bar{\eta}$  is an equivariant hermitian vector bundle on  $Y$  and that  $\bar{\xi}_\bullet$  is a complex of equivariant hermitian vector bundles providing a resolution of  $i_*\bar{\eta}$  on  $X$  whose metrics satisfy Bismut assumption (A). Then, fixing an element  $g \in G$ , J.-M. Bismut can construct a singular current  $T_g(\bar{\xi}_\bullet) \in D(X_g)$  which is a sum of  $(p, p)$ -type currents satisfying

(i'). (Differential equation)  $\text{dd}^c T_g(\bar{\xi}_\bullet) = i_{g*}(\text{ch}_g(\bar{\eta}) \text{Td}_g^{-1}(\bar{N})) - \sum_k (-1)^k \text{ch}_g(\bar{\xi}_k)$ .

As in the case of Bott-Chern secondary characteristic classes, it can be shown that  $T_g(\bar{\xi}_\bullet)$  also satisfies other two properties

(ii'). (Functoriality)  $f_g^* T_g(\bar{\xi}_\bullet) = T_g(f_g^* \bar{\xi}_\bullet)$  if  $f : X' \rightarrow X$  is a  $G$ -equivariant holomorphic map of complex manifolds which is transversal to  $Y$ .

(iii') (Normalization)  $T_g(\bar{\xi}_\bullet) = -\tilde{\text{ch}}_g(\bar{\xi}_\bullet)$  if  $Y$  is the empty set.

Naturally, one hopes that such three properties are enough to characterize a theory of equivariant singular Bott-Chern classes. But unfortunately this is not true,  $T_g(\bar{\xi}_\bullet)$  is not the unique element which satisfies the properties (i'), (ii') and (iii') even in the current class space  $\tilde{\mathcal{U}}(X_g)$ .

For the non-equivariant case i.e. when  $G$  is the trivial group, J. I. Burgos Gil and R. Lițcanu have obtained a satisfactory axiomatic characterization of singular Bott-Chern classes in their article [5]. They realized this by adding a natural fourth axiom to the properties (i'), (ii') and (iii') (removing the subscript  $g$ ) which is called the condition of homogeneity. In this paper, we will do the equivariant version. Our strategy is basically the same as that was used in J. I. Burgos Gil and R. Lițcanu's article. By deforming a resolution to a easily understandable one, we show that a theory of equivariant singular Bott-Chern classes is totally determined by its effects on Koszul resolutions (cf. Theorem 6.1). This approach can be viewed as an analogue of J.-M. Bismut, H. Gillet and C. Soulé's axiomatic construction of Bott-Chern secondary characteristic classes.

The original purpose of the author's study of the uniqueness property of equivariant singular Bott-Chern classes is that he wants to prove a purely analytic statement which is called the concentration formula. Such a formula plays a crucial role in the proof of an arithmetic concen-

tration theorem in Arakelov geometry. We shall formulate this result in the last section of this paper.

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## 2 Equivariant secondary characteristic classes

To every hermitian vector bundle on a compact complex manifold we can associate a smooth differential form by using Chern-Weil theory. Notice that Chern-Weil theory is not additive for short exact sequence of hermitian vector bundles, the Bott-Chern secondary characteristic classes cover this gap. In this section, we shall recall how to generalize all these things above to the equivariant setting, namely for a compact complex manifold  $X$  which admits a holomorphic action of a compact Lie group  $G$ .

Let  $g \in G$  be an automorphism of  $X$ , we denote by  $X_g = \{x \in X \mid g \cdot x = x\}$  the fixed point submanifold.  $X_g$  is also a compact complex manifold. Let  $\overline{E}$  be an equivariant hermitian vector bundle on  $X$ , this means that  $E$  admits a  $G$ -action which is compatible with the  $G$ -action on  $X$  and that the metric on  $E$  is invariant under the action of  $G$ . If there is no additional description, a morphism between equivariant vector bundles will be a morphism of vector bundles which is compatible with the equivariant structures. It is well known that the restriction of an equivariant hermitian vector bundle  $\overline{E}$  to  $X_g$  splits as a direct sum

$$\overline{E}|_{X_g} = \bigoplus_{\zeta \in S^1} \overline{E}_\zeta$$

where the equivariant structure  $g^E$  of  $E$  acts on  $\overline{E}_\zeta$  as  $\zeta$ . We often write  $\overline{E}_g$  for  $\overline{E}_1$  and denote its orthogonal complement by  $\overline{E}_\perp$ . As usual,  $A^{p,q}(X)$  stands for the space of  $(p, q)$ -forms  $\Gamma^\infty(X, \Lambda^p T^{*(1,0)} X \wedge \Lambda^q T^{*(0,1)} X)$ , we define

$$\tilde{A}(X) = \bigoplus_{p=0}^{\dim X} (A^{p,p}(X) / (\text{Im } \partial + \text{Im } \overline{\partial})).$$

We denote by  $\Omega^{\overline{E}_\zeta}$  the curvature matrix associated to  $\overline{E}_\zeta$ . Let  $(\phi_\zeta)_{\zeta \in S^1}$  be a family of  $\mathbf{GL}(\mathbb{C})$ -invariant formal power series such that  $\phi_\zeta \in \mathbb{C}[[\mathbf{gl}_{\text{rk } E_\zeta}(\mathbb{C})]]$  where  $\text{rk } E_\zeta$  stands for the rank of  $E_\zeta$  which is a locally constant function on  $X_g$ . Moreover, let  $\phi \in \mathbb{C}[[\bigoplus_{\zeta \in S^1} \mathbb{C}]]$  be any formal power series. We have the following definition.

**Definition 2.1.** The way to associate a smooth form to an equivariant hermitian vector bundle  $\overline{E}$  by setting

$$\phi_g(\overline{E}) := \phi((\phi_\zeta(-\frac{\Omega^{\overline{E}_\zeta}}{2\pi i}))_{\zeta \in S^1})$$

is called an equivariant Chern-Weil theory associated to  $(\phi_\zeta)_{\zeta \in S^1}$  and  $\phi$ . The class of  $\phi_g(\overline{E})$  in  $\tilde{A}(X_g)$  is independent of the metric.

The theory of equivariant secondary characteristic classes is described in the following theorem.

**Theorem 2.2.** *To every short exact sequence  $\overline{\varepsilon} : 0 \rightarrow \overline{E}' \rightarrow \overline{E} \rightarrow \overline{E}'' \rightarrow 0$  of equivariant hermitian vector bundles on  $X$ , there is a unique way to attach a class  $\tilde{\phi}_g(\overline{\varepsilon}) \in \tilde{A}(X_g)$  which satisfies the following three conditions:*

(i).  $\tilde{\phi}_g(\overline{\varepsilon})$  satisfies the differential equation

$$\mathrm{dd}^c \tilde{\phi}_g(\overline{\varepsilon}) = \phi_g(\overline{E}' \oplus \overline{E}'') - \phi_g(\overline{E});$$

(ii). for every equivariant holomorphic map  $f : X' \rightarrow X$ ,  $\tilde{\phi}_g(f^* \overline{\varepsilon}) = f_g^* \tilde{\phi}_g(\overline{\varepsilon})$ ;

(iii).  $\tilde{\phi}_g(\overline{\varepsilon}) = 0$  if  $\overline{\varepsilon}$  is equivariantly and orthogonally split.

*Proof.* Firstly note that one can carry out the principle of [1, Section f.] to construct a new exact sequence of equivariant hermitian vector bundles

$$\tilde{\varepsilon} : 0 \rightarrow \overline{E}'(1) \rightarrow \tilde{\overline{E}} \rightarrow \overline{E}'' \rightarrow 0$$

on  $X \times \mathbb{P}^1$  such that  $i_0^* \tilde{\varepsilon}$  is isometric to  $\overline{\varepsilon}$  and  $i_\infty^* \tilde{\varepsilon}$  is equivariantly and orthogonally split. Here the projective line  $\mathbb{P}^1$  carries the trivial  $G$ -action and the section  $i_0$  (resp.  $i_\infty$ ) is defined by setting  $i_0(x) = (x, 0)$  (resp.  $i_\infty(x) = (x, \infty)$ ). Then one can show that an equivariant secondary characteristic class  $\tilde{\phi}_g(\tilde{\varepsilon})$  which satisfies the three conditions in the statement of this theorem must be of the form

$$\tilde{\phi}_g(\tilde{\varepsilon}) = - \int_{\mathbb{P}^1} \phi_g(\tilde{E}, h^{\tilde{E}}) \cdot \log |z|^2.$$

So the uniqueness has been proved. For the existence, one may take this identity as the definition of the equivariant secondary class  $\tilde{\phi}_g$ , of course one should verify that this definition is independent of the choice of the metric  $h^{\tilde{E}}$  and really satisfies the three conditions above. The verification is totally the same as the non-equivariant case, one just add the subscript  $g$  to every corresponding notation.

Another way to show the existence is to use the non-equivariant secondary classes on  $X_g$  directly. We first split  $\overline{\varepsilon}$  on  $X_g$  into a family of short exact sequences

$$\overline{\varepsilon}_\zeta : 0 \rightarrow \overline{E}'_\zeta \rightarrow \overline{E}_\zeta \rightarrow \overline{E}''_\zeta \rightarrow 0$$

for all  $\zeta \in S^1$ . Using the non-equivariant secondary classes on  $X_g$  we define for  $\zeta, \eta \in S^1$

$$\widetilde{(\phi_\zeta + \phi_\eta)}(\overline{\varepsilon}_\zeta, \overline{\varepsilon}_\eta) := \tilde{\phi}_\zeta(\overline{\varepsilon}_\zeta) + \tilde{\phi}_\eta(\overline{\varepsilon}_\eta)$$

and

$$\widetilde{(\phi_\zeta \cdot \phi_\eta)}(\overline{\varepsilon}_\zeta, \overline{\varepsilon}_\eta) := \tilde{\phi}_\zeta(\overline{\varepsilon}_\zeta) \cdot \phi_\eta(\overline{E}_\eta) + \phi_\zeta(\overline{E}'_\zeta + \overline{E}''_\zeta) \cdot \tilde{\phi}_\eta(\overline{\varepsilon}_\eta)$$

and similarly for other finite sums and products. With these notations we define  $\tilde{\phi}_g(\bar{\varepsilon}) := \phi(\widetilde{(\phi_\zeta)_{\zeta \in S^1}})((\bar{\varepsilon}_\zeta)_{\zeta \in S^1})$ . The equivariant secondary class  $\tilde{\phi}_g$  defined like this way satisfies the three conditions in the statement of this theorem, this fact follows from the axiomatic characterization of non-equivariant secondary classes.  $\square$

**Remark 2.3.** (i). The first way to construct equivariant secondary characteristic classes is also valid for long exact sequences of hermitian vector bundles  $\bar{\varepsilon} : 0 \rightarrow \bar{E}_m \rightarrow \bar{E}_{m-1} \rightarrow \cdots \rightarrow \bar{E}_1 \rightarrow \bar{E}_0 \rightarrow 0$ . Here the sign is chosen so that

$$\mathrm{dd}^c \tilde{\phi}_g(\bar{\varepsilon}) = \phi_g\left(\bigoplus_{j \text{ even}} \bar{E}_j\right) - \phi_g\left(\bigoplus_{j \text{ odd}} \bar{E}_j\right).$$

That means there exists an exact sequence  $\tilde{\bar{\varepsilon}}$  on  $X \times \mathbb{P}^1$  such that  $i_0^* \tilde{\bar{\varepsilon}}$  is isometric to  $\bar{\varepsilon}$  and  $i_\infty^* \tilde{\bar{\varepsilon}}$  is equivariantly and orthogonally split. This new exact sequence is called the first transgression exact sequence of  $\bar{\varepsilon}$  and will be denoted by  $\mathrm{tr}_1(\bar{\varepsilon})$ .

(ii). The first part of this remark gives a uniqueness theorem for secondary classes for long exact sequences. Then when  $\phi_g$  is additive one can have another way to construct the secondary classes, that is to split a long exact sequence into a series of short exact sequences and use the secondary classes in Theorem 2.2 to formulate an alternating sum. This alternating sum provides a secondary class for original long exact sequence.

We now give some examples of equivariant character forms and their corresponding secondary characteristic classes.

**Example 2.4.** (i). The equivariant Chern character form  $\mathrm{ch}_g(\bar{E}) := \sum_{\zeta \in S^1} \zeta \mathrm{ch}(\bar{E}_\zeta)$ .

(ii). The equivariant Todd form  $\mathrm{Td}_g(\bar{E}) := \frac{c_{\mathrm{rk} E_g}(\bar{E}_g)}{\mathrm{ch}_g(\sum_{j=0}^{\mathrm{rk} E} (-1)^j \wedge^j \bar{E}^\vee)}$ . As in [11, Thm. 10.1.1] one can show that

$$\mathrm{Td}_g(\bar{E}) = \mathrm{Td}(\bar{E}_g) \prod_{\zeta \neq 1} \det\left(\frac{1}{1 - \zeta^{-1} e^{\frac{\bar{E}_\zeta}{2\pi i}}}\right).$$

(iii). Let  $\bar{\varepsilon} : 0 \rightarrow \bar{E}' \rightarrow \bar{E} \rightarrow \bar{E}'' \rightarrow 0$  be a short exact sequence of hermitian vector bundles. The secondary Bott-Chern characteristic class is given by  $\tilde{\mathrm{ch}}_g(\bar{\varepsilon}) = \sum_{\zeta \in S^1} \zeta \tilde{\mathrm{ch}}(\bar{\varepsilon}_\zeta)$ .

(iv). If the equivariant structure  $g^\varepsilon$  has the eigenvalues  $\zeta_1, \dots, \zeta_m$ , then the secondary Todd class is given by

$$\widetilde{\mathrm{Td}}_g(\bar{\varepsilon}) = \sum_{i=1}^m \prod_{j=1}^{i-1} \mathrm{Td}_g(\bar{E}_{\zeta_j}) \cdot \widetilde{\mathrm{Td}}(\bar{\varepsilon}_{\zeta_i}) \cdot \prod_{j=i+1}^m \mathrm{Td}_g(\bar{E}'_{\zeta_j} + \bar{E}''_{\zeta_j}).$$

**Remark 2.5.** One can use Theorem 2.2 to give a proof of the statements (iii) and (iv) in the examples above.

**Lemma 2.6.** Let  $\bar{\varepsilon}$  be an acyclic complex of equivariant hermitian vector bundles. Then for any non-negative integer  $k$  we have

$$\tilde{\phi}_g(\bar{\varepsilon}[-k]) = (-1)^k \tilde{\phi}_g(\bar{\varepsilon}).$$

*Proof.* This follows from the construction of  $\bar{\varepsilon}[-k]$  which is obtained by shifting degree.  $\square$

A particular secondary class when we consider a fixed vector bundle with two different hermitian metrics will be used frequently in our paper, so we formulate it separately in the following definition.

**Definition 2.7.** Let  $E$  be an equivariant vector bundle on  $X$ . Assume that  $h_0$  and  $h_1$  are two invariant hermitian metrics on  $E$ . We denote by  $\tilde{\phi}_g(E, h_0, h_1)$  the equivariant secondary characteristic class associated to the short exact sequence

$$0 \rightarrow 0 \rightarrow (E, h_1) \rightarrow (E, h_0) \rightarrow 0$$

so that we have the differential equation  $\text{dd}^c \tilde{\phi}_g(E, h_0, h_1) = \phi_g(E, h_0) - \phi_g(E, h_1)$ .

The following proposition describes the additivity of equivariant secondary characteristic classes.

**Proposition 2.8.** *Let*

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bar{E}'_1 & \longrightarrow & \bar{E}_1 & \longrightarrow & \bar{E}''_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bar{E}'_2 & \longrightarrow & \bar{E}_2 & \longrightarrow & \bar{E}''_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bar{E}'_3 & \longrightarrow & \bar{E}_3 & \longrightarrow & \bar{E}''_3 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

be a double complex of equivariant hermitian vector bundles on  $X$  where all rows  $\bar{\varepsilon}_i$  and all columns  $\bar{\delta}_j$  are exact. Then we have

$$\tilde{\phi}_g(\bar{\varepsilon}_1 \oplus \bar{\varepsilon}_3) - \tilde{\phi}_g(\bar{\varepsilon}_2) = \tilde{\phi}_g(\bar{\delta}_1 \oplus \bar{\delta}_3) - \tilde{\phi}_g(\bar{\delta}_2).$$

*Proof.* We may have the corresponding diagram of hermitian vector bundles on  $X \times \mathbb{P}^1$  by the first construction in the proof of Theorem 2.2. Then

$$\begin{aligned} \tilde{\phi}_g(\bar{\varepsilon}_2) - \tilde{\phi}_g(\bar{\varepsilon}_1 \oplus \bar{\varepsilon}_3) &= - \int_{\mathbb{P}^1} [\phi_g(\widetilde{E}_2, h^{\widetilde{E}_2}) - \phi_g(\widetilde{E}_1 \oplus \widetilde{E}_3, h^{\widetilde{E}_1} \oplus h^{\widetilde{E}_3})] \cdot \log |z|^2 \\ &= \int_{\mathbb{P}^1} \text{dd}^c \tilde{\phi}_g(\bar{\delta}_2) \cdot \log |z|^2 = \int_{\mathbb{P}^1} \tilde{\phi}_g(\bar{\delta}_2) \cdot \text{dd}^c \log |z|^2 \\ &= i_0^* \tilde{\phi}_g(\bar{\delta}_2) - i_\infty^* \tilde{\phi}_g(\bar{\delta}_2) = \tilde{\phi}_g(\bar{\delta}_2) - \tilde{\phi}_g(\bar{\delta}_1 \oplus \bar{\delta}_3). \end{aligned}$$

$\square$

**Remark 2.9.** This proposition can be generalized without any difficulty to the case of a bounded exact sequence of bounded exact sequences of equivariant hermitian vector bundles. Let  $\overline{A}_{*,*}$  be such a double acyclic complex, we have

$$\tilde{\phi}_g\left(\bigoplus_{k \text{ even}} \overline{A}_{k,*}\right) - \tilde{\phi}_g\left(\bigoplus_{k \text{ odd}} \overline{A}_{k,*}\right) = \tilde{\phi}_g\left(\bigoplus_{k \text{ even}} \overline{A}_{*,k}\right) - \tilde{\phi}_g\left(\bigoplus_{k \text{ odd}} \overline{A}_{*,k}\right).$$

**Corollary 2.10.** Let  $\overline{A}_{*,*}$  be a bounded double complex of equivariant hermitian vector bundles with exact rows and exact columns, then we have

$$\tilde{\phi}_g(\text{Tot} \overline{A}_{*,*}) = \tilde{\phi}_g\left(\bigoplus_k \overline{A}_{k,*}[-k]\right).$$

*Proof.* For any non-negative integer  $n$  we denote by  $\text{Tot}_n = \text{Tot}((\overline{A}_{k,*})_{k \geq n})$  the total complex of the exact complex formed by the rows with index bigger than  $n-1$ . Then  $\text{Tot}_0 = \text{Tot}(\overline{A}_{*,*})$ . By an argument of induction, for each  $k \geq 0$  we have an exact sequence of complexes

$$0 \rightarrow \text{Tot}_{k+1} \rightarrow \text{Tot}_k \oplus \bigoplus_{l < k} \overline{A}_{l,*}[-l] \rightarrow \bigoplus_{l \leq k} \overline{A}_{l,*}[-l] \rightarrow 0$$

which is orthogonally split in each degree. Therefore by Proposition 2.8 and Remark 2.9 we get

$$\tilde{\phi}_g(\text{Tot}_k \oplus \bigoplus_{l < k} \overline{A}_{l,*}[-l]) = \tilde{\phi}_g(\text{Tot}_{k+1} \oplus \bigoplus_{l \leq k} \overline{A}_{l,*}[-l]).$$

By induction, when  $k$  is taken to be big enough we prove the statement.  $\square$

### 3 Cohomology of currents with fixed wave front sets

This section is devoted to recall the results of [5, Section 4] and to derive some standard consequences. To be more precise, we recall that there is a classic theorem concerning the complex of currents on a compact complex manifold which says that its cohomology groups are isomorphic to the cohomology groups of the complex of smooth forms. In this section, we shall prove a similar theorem for the currents with any fixed wave front set. This theorem implies a certain  $\partial\bar{\partial}$ -lemma.

Let  $X$  be a compact complex manifold of dimension  $d$ . Then the space  $A^n(X)$  of  $C^\infty$  complex valued  $n$ -forms on  $X$  is a topological vector space with Schwartz topology (cf. [16, Chapter IX]). We denote by  $D_n(X)$  the continuous dual of  $A^n(X)$  which is called the space of currents of dimension  $n$  on  $X$ . Note that  $X$  is a complex manifold, we have the following decomposition

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X)$$

and Dolbeault operators  $\partial : A^{p,q}(X) \rightarrow A^{p+1,q}(X)$ ,  $\bar{\partial} : A^{p,q}(X) \rightarrow A^{p,q+1}(X)$  with  $d = \partial + \bar{\partial}$  from  $A^n(X)$  to  $A^{n+1}(X)$  the usual differentials.

All things above induce corresponding notations for  $D_n(X)$  as follows

$$D_n(X) = \bigoplus_{p+q=n} D_{p,q}(X)$$

and Dolbeault operators  $\partial' : D_{p+1,q}(X) \rightarrow D_{p,q}(X)$ ,  $\bar{\partial}' : D_{p,q+1}(X) \rightarrow D_{p,q}(X)$  with  $d' = \partial' + \bar{\partial}'$  from  $D_{n+1}(X)$  to  $D_n(X)$ . Here the differential  $d'$  should be understood as for any  $T \in D_{n+1}(X)$ ,  $\alpha \in A^n(X)$ ,  $d'T(\alpha) = T(d\alpha)$ . We now give two basic examples of currents which will be used frequently.

**Example 3.1.** If  $i : Y \hookrightarrow X$  is a  $k$ -dimensional analytic subspace of  $X$ , we may define a  $2k$ -dimensional current  $\delta_Y$  which is introduced by Lelong [15] by

$$\delta_Y(\alpha) = \int_{Y^{ns}} i^* \alpha, \quad \alpha \in A^{2k}(X)$$

where  $Y^{ns}$  is the subset of non-singular points in  $Y$ . Note that  $\delta_Y$  actually belongs to  $D_{k,k}(X)$  since if  $\alpha^{p,q} \in A^{p,q}(X)$  with  $p+q=2k$ , then  $i^* \alpha = 0$  unless  $p=q=k$ .

**Example 3.2.** We may have the following products

$$D_n(X) \otimes A^m(X) \rightarrow D_{n-m}(X)$$

which decompose into

$$D_{p,q}(X) \otimes A^{r,s}(X) \rightarrow D_{p-r,q-s}(X).$$

Actually for  $T \in D_n(X)$ ,  $\alpha \in A^m(X)$ , we denote their product by  $T \wedge \alpha$ , and if  $\beta \in A^{n-m}(X)$ , the product is defined by

$$(T \wedge \alpha)(\beta) = T(\alpha \wedge \beta).$$

In particular, we have a map from  $A^{p,q}(X)$  to  $D_{d-p,d-q}(X)$  which maps  $\alpha$  to  $\delta_X \wedge \alpha$ . We often write  $\delta_X \wedge \alpha$  as  $[\alpha]$  for simplicity. From [16, Chapter X] we know that the spaces  $D_{p,q}$  have a natural topology, for which the maps  $A^{p,q}(X) \rightarrow D_{d-p,d-q}(X)$  are continuous with dense images. So if we write  $D^{p,q}(X) = D_{d-p,d-q}(X)$ , we may have the following embedding

$$A^{p,q}(X) \hookrightarrow D^{p,q}(X).$$

We would like to indicate that, more generally, if  $\alpha$  is a  $L^1$ -form i.e.  $\alpha$  has coefficients which are locally integrable then  $[\alpha]$  is a well-defined current.

**Remark 3.3.** The map  $\alpha \mapsto [\alpha]$  doesn't send  $d$  to  $d'$ . In fact, for  $\alpha \in A^n(X)$  and  $\beta \in A^{d-n-1}(X)$ , by Stokes theorem we will have

$$\begin{aligned} [d\alpha](\beta) &= \int_X d\alpha \wedge \beta = \int_X d(\alpha \wedge \beta) - \int_X (-1)^n \alpha \wedge d\beta \\ &= (-1)^{n+1} \int_X \alpha \wedge d\beta = (-1)^{n+1} (d'[\alpha])(\beta). \end{aligned}$$



So if we write  $d = (-1)^{n+1}d'$  the differential from  $D^n(X)$  to  $D^{n+1}(X)$ , then the inclusion  $A^n(X) \hookrightarrow D^n(X)$  commutes with  $d$ . The same conclusions can be obtained for  $\partial$  and  $\bar{\partial}$ . And one should notice that this commutativity induces a family of morphisms at the level of cohomology with respect to  $\partial, \bar{\partial}$  and  $d$ . These morphisms are actually isomorphisms.

The wave front set  $\text{WF}(\eta)$  of a current  $\eta$  is a closed conical subset of  $T_{\mathbb{R}}^*X_0 := T_{\mathbb{R}}^*X \setminus \{0\}$ , the real cotangent bundle removed the complete zero section. This conical subset measures the singularities of  $\eta$ , actually the projection of  $\text{WF}(\eta)$  in  $X$  is equal to the singular locus of the support of  $\eta$ . It also allows us to define certain products and pull-backs of currents. Let  $S$  be a conical subset of  $T_{\mathbb{R}}^*X_0$  and let  $D^*(X, S)$  stand for the spaces consisting of all currents whose wave front sets are contained in  $S$ . Now suppose that  $P$  is a differential operator with smooth coefficients, then we have  $\text{WF}(P \circ \eta) \subseteq \text{WF}(\eta)$  by [12, (8.1.11)]. This means  $D^*(X, S)$  form a  $\partial$ -,  $\bar{\partial}$ - or  $d$ -complex.

Let  $f : Y \rightarrow X$  be a morphism of compact complex manifolds. The set of normal directions of  $f$  is

$$N_f = \{(f(y), v) \in T_{\mathbb{R}}^*X \mid df^t v = 0\}.$$

This set measures the singularities of the morphism  $f$ . Actually, if  $f$  is smooth then  $N_f = 0$  and if  $f$  is a closed immersion then  $N_f$  is the conormal bundle  $N_{X/Y, \mathbb{R}}^\vee$ . Let  $S \subset T_{\mathbb{R}}^*X_0$  be a closed conical subset, the morphism  $f$  is transversal to  $S$  if  $N_f \cap S = \emptyset$ .

**Theorem 3.4.** *Let  $f : Y \rightarrow X$  be a morphism of compact complex manifolds which is transversal to  $S$ . Then there is a unique way to extend the pull-back  $f^* : A^*(X) \rightarrow A^*(Y)$  to a continuous morphism of complexes*

$$f^* : D^*(X, S) \rightarrow D^*(Y, f^*S).$$

*Proof.* This follows from [12, Theorem 8.2.4]. Here the topology on current space  $D^*$  is given by [12, Definition 8.2.2] which is finer than the usual dual topology.  $\square$

**Theorem 3.5.** ( $\bar{\partial}$ -Poincaré lemma) *For any integer  $p \geq 0$ , denote by  $\mathcal{D}_{X,S}^{p,*}$  the sheaf of currents of type  $(p, *)$  whose wave front sets are contained in  $S$ . Then for each  $q > 0$ , any  $\bar{\partial}$ -closed section of  $\mathcal{D}_{X,S}^{p,q}$  is locally  $\bar{\partial}$ -exact.*

*Proof.* Let  $\mathcal{A}_X^{p,*}$  be the sheaf of smooth forms of type  $(p, *)$  on  $X$ , we claim that the natural inclusions

$$\iota : \mathcal{A}_X^{p,q} \rightarrow \mathcal{D}_{X,S}^{p,q}$$

induce a quasi-isomorphism between complexes. Actually, this claim is the content of [5, Theorem 4.5]. With this observation we may reduce our problem to the case of smooth forms which is classical, one can find in [8, Page 25] a proof of this statement.  $\square$

**Corollary 3.6.** *Let notations and assumptions be as in Theorem 3.5 and its proof, then the natural morphisms  $H^{p,*}(\mathcal{A}_X^{p,*}(X), \bar{\partial}) \longrightarrow H^{p,*}(\mathcal{D}_{X,S}^{p,*}(X), \bar{\partial})$  are isomorphisms.*

*Proof.* We denote by  $\Omega^p$  the sheaf of holomorphic  $p$ -forms on  $X$ . The Dolbeault theorem says that  $H^*(X, \Omega^p)$  are isomorphic to  $H^{p,*}(\mathcal{A}_X^{p,*}(X), \bar{\partial})$ . The proof of the Dolbeault theorem relies on two deep results, one is that the following complex of sheaves

$$0 \longrightarrow \Omega^p \longrightarrow \mathcal{A}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,d} \longrightarrow 0$$

is exact, the other one is that the sheaves  $\mathcal{A}_X^{p,*}$  admit partitions of unity so that  $H^k(X, \mathcal{A}_X^{p,*}) = 0$  for  $k > 0$ . Note that the sheaves  $\mathcal{D}_{X,S}^{p,*}$  may be multiplied by  $C^\infty$  functions, hence they also admit partitions of unity. Therefore one can carry out the principle of the sheaf-theoretic proof of Dolbeault theorem to prove that  $H^*(X, \Omega^p) \cong H^{p,*}(\mathcal{D}_{X,S}^{p,*}(X), \bar{\partial})$  if the following complex of sheaves

$$0 \longrightarrow \Omega^p \longrightarrow \mathcal{D}_{X,S}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{D}_{X,S}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{D}_{X,S}^{p,d} \longrightarrow 0$$

is exact. Such a Dolbeault theorem for currents implies our statements in this corollary. Indeed, the complex of sheaves above is really exact, the exactness at 0-degree is just the regularity theorem for the  $\bar{\partial}$ -operator (cf. [8, Page 380]) and the exactness at higher degrees is implied by the  $\bar{\partial}$ -Poincaré lemma, Theorem 3.5.  $\square$

**Remark 3.7.** One can prove the similar results for  $\partial$ -cohomology and de Rham cohomology, namely the natural morphisms  $H^{*,p}(\mathcal{A}_X^{*,p}(X), \partial) \longrightarrow H^{*,p}(\mathcal{D}_{X,S}^{*,p}(X), \partial)$  and  $H_{\text{DR}}^*(X) \longrightarrow H^*(\mathcal{D}_{X,S}^*(X), d)$  are all isomorphisms.

**Corollary 3.8.** *Let  $\mathcal{D}_X^{p,*}$  be the sheaf of currents of type  $(p, *)$  on  $X$ , then the natural morphisms  $H^{p,*}(\mathcal{D}_{X,S}^{p,*}(X), \bar{\partial}) \longrightarrow H^{p,*}(\mathcal{D}_X^{p,*}(X), \bar{\partial})$  are isomorphisms.*

*Proof.* This follows from Corollary 3.6.  $\square$

This Corollary implies the following  $\partial\bar{\partial}$ -lemma.

**Theorem 3.9.** *Let  $X$  be a compact complex manifold and let  $S$  be a closed conical subset of  $T_{\mathbb{R}}^*X_0$ . Then:*

(i). *If  $\gamma$  is a current on  $X$  such that  $\partial\bar{\partial}\gamma \in D^*(X, S)$ , then there exist currents  $\alpha$  and  $\beta$  such that  $\gamma = \omega + \partial\alpha + \bar{\partial}\beta$  with  $\omega \in D^*(X, S)$ .*

(ii). *If  $\omega$  is an element in  $D^*(X, S)$  such that  $\omega = \partial u + \bar{\partial}v$  for currents  $u$  and  $v$ , then there exist currents  $\alpha, \beta \in D^*(X, S)$  such that  $\omega = \partial\alpha + \bar{\partial}\beta$ .*

*Proof.* (i). The hypothesis  $\partial\bar{\partial}\gamma = \eta$  with  $\eta \in D^*(X, S)$  implies that  $\eta = \partial(\bar{\partial}\gamma)$  and hence  $\eta = \partial\alpha$  for some  $\alpha \in D^*(X, S)$ . So  $\partial(\bar{\partial}\gamma - \alpha) = 0$  and  $\bar{\partial}\gamma - \alpha = \beta + \partial\gamma_1$  with  $\beta \in D^*(X, S)$ . So we know that  $\partial\bar{\partial}\gamma_1 = \eta_1 = \bar{\partial}(\alpha + \beta)$  is contained in  $D^*(X, S)$ . By repeating this argument we get a sequence of currents  $\gamma_n$  such that  $\bar{\partial}\gamma_n = u_n + \partial\gamma_{n+1}$  with  $u_n \in D^*(X, S)$ .

Note that if we assume that  $\gamma_n \in D^{p,q}(X)$ , then  $\gamma_{n+1}$  should be in  $D^{p-1,q+1}(X)$  by construction. So when  $n$  is big enough we have  $\gamma_{n+1} = 0$ . Therefore  $\bar{\partial}\gamma_n = u_n$  is contained in  $D^*(X, S)$ , hence  $\gamma_n = \omega_n + \bar{\partial}\beta_n$  with  $\omega_n \in D^*(X, S)$ . So  $\bar{\partial}(\gamma_{n-1} + \partial\beta_n) = u_{n-1} + \partial\omega_n$  is contained in

$D^*(X, S)$ , and therefore  $\gamma_{n-1} = \omega_{n-1} + \partial\alpha_{n-1} + \bar{\partial}\beta_{n-1}$  with  $\omega_{n-1} \in D^*(X, S)$ . By repeating this argument we are done.

(ii). If  $\omega = \partial u + \bar{\partial}v$ , then  $\partial\omega = \partial\bar{\partial}v$  which implies that  $v = \alpha + \partial x + \bar{\partial}y$  with  $\alpha \in D^*(X, S)$  by (i). So we have  $\bar{\partial}v = \bar{\partial}\alpha + \bar{\partial}\partial x$ . Similarly  $\partial u = \partial\beta + \partial\bar{\partial}z$  with  $\beta \in D^*(X, S)$ . Therefore  $\omega = \partial\alpha + \bar{\partial}\beta + \partial\bar{\partial}(z-x)$ . Again by (i),  $z-x = \gamma + \partial s + \bar{\partial}t$  with  $\gamma \in D^*(X, S)$ . So  $\partial\bar{\partial}(z-x) = \partial\bar{\partial}\gamma$  which implies that  $\omega = \partial(\alpha + \bar{\partial}\gamma) + \bar{\partial}\beta$ .  $\square$

## 4 Deformation to the normal cone

By a projective manifold we shall understand a compact complex manifold which is projective algebraic, that means a projective manifold is the complex analytic space  $X(\mathbb{C})$  associated to a smooth projective variety  $X$  over  $\mathbb{C}$ . Denote by  $\mu_n$  the diagonalisable group variety over  $\mathbb{C}$  associated to  $\mathbb{Z}/n\mathbb{Z}$ , we say  $X$  is equivariant if it admits a  $\mu_n$ -projective action (cf. [14, Section 2]). Write  $X_{\mu_n}$  for the fixed point subscheme, by GAGA principle,  $X_{\mu_n}(\mathbb{C})$  is equal to  $X(\mathbb{C})_g$  where  $g$  is the automorphism on  $X(\mathbb{C})$  corresponding to a fixed primitive  $n$ -th root of unity. From now on, if no confusion arises, we shall not distinguish between  $X$  and  $X(\mathbb{C})$  as well as  $X_{\mu_n}$  and  $X_g$ .

In this section, we shall describe the algebro-geometric preliminaries for the discussion of the uniqueness of equivariant singular Bott-Chern classes. Our main tool is an elegant method so called the deformation to the normal cone which allows us to deform a resolution of hermitian vector bundle associated to a closed immersion of projective manifolds to a simpler one. This will help us to formulate the analytic data (e.g. the secondary characteristic class) of the original resolution by using the corresponding analytic data of the new one. This process is just like the first construction we mentioned in the proof of Theorem 2.2.

The first part of this section is devoted to recall the deformation to the normal cone technique which can be found in several standard literatures, for example in [2, Section 4]. The second part is devoted to the equivariant analogue.

Let  $i : Y \hookrightarrow X$  be a closed immersion of projective manifolds. We will denote by  $N_{X/Y}$  the normal bundle of this immersion. For a vector bundle  $E$  on  $X$  or  $Y$ , the notation  $\mathbb{P}(E)$  will stand for the projective space bundle  $\text{Proj}(\text{Sym}(E^\vee))$ .

**Definition 4.1.** The deformation to the normal cone  $W(i)$  of the immersion  $i$  is the blowing up of  $X \times \mathbb{P}^1$  along  $Y \times \{\infty\}$ . We shall just write  $W$  for  $W(i)$  if there is no confusion about the immersion.

We denote by  $p_X$  (resp.  $p_Y$ ) the projection  $X \times \mathbb{P}^1 \rightarrow X$  (resp.  $Y \times \mathbb{P}^1 \rightarrow Y$ ) and by  $\pi$  the blow-down map  $W \rightarrow X \times \mathbb{P}^1$ . We also denote by  $q_X$  (resp.  $q_Y$ ) the projection  $X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  (resp.  $Y \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ ) and by  $q_W$  the composition  $q_X \circ \pi$ . It is well known that the map  $q_W$  is flat and for  $t \in \mathbb{P}^1$ , we have

$$q_W^{-1}(t) \cong \begin{cases} X \times \{t\}, & \text{if } t \neq \infty, \\ P \cup \tilde{X}, & \text{if } t = \infty, \end{cases}$$

where  $\tilde{X}$  is isomorphic to the blowing up of  $X$  along  $Y$  and  $P$  is isomorphic to the projective completion of  $N_{X/Y}$  i.e. the projective space bundle  $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ . Denote the canonical projection from  $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$  to  $Y$  by  $\pi_P$ , then the morphism  $\mathcal{O}_Y \rightarrow N_{X/Y} \oplus \mathcal{O}_Y$  induces a canonical section  $i_\infty : Y \hookrightarrow \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$  which is called the zero section embedding. Moreover, let  $j : Y \times \mathbb{P}^1 \rightarrow W$  be the canonical closed immersion induced by  $i \times \text{Id}$ , then the component  $\tilde{X}$  doesn't meet  $j(Y \times \mathbb{P}^1)$  and the intersection of  $j(Y \times \mathbb{P}^1)$  and  $P$  is exactly the image of  $Y$  under the section  $i_\infty$ .

On  $P = \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ , there exists a tautological exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \pi_P^*(N_{X/Y} \oplus \mathcal{O}_Y) \rightarrow Q \rightarrow 0$$

where  $Q$  is the tautological quotient bundle. This exact sequence and the inclusion  $\mathcal{O}_P \rightarrow \pi_P^*(N_{X/Y} \oplus \mathcal{O}_Y)$  induce a section  $\sigma : \mathcal{O}_P \rightarrow Q$  which vanishes along the zero section  $i_\infty(Y)$ . By duality we get a morphism  $Q^\vee \rightarrow \mathcal{O}_P$ , and this morphism induces the following exact sequence

$$0 \rightarrow \wedge^n Q^\vee \rightarrow \cdots \rightarrow \wedge^2 Q^\vee \rightarrow Q^\vee \rightarrow \mathcal{O}_P \rightarrow i_{\infty*} \mathcal{O}_Y \rightarrow 0$$

where  $n$  is the rank of  $Q$ . Note that  $i_\infty$  is a section of  $\pi_P$  i.e.  $\pi_P \circ i_\infty = \text{Id}$ , the projection formula implies the following definition.

**Definition 4.2.** For any vector bundle  $F$  on  $Y$ , the following complex of vector bundles

$$0 \rightarrow \wedge^n Q^\vee \otimes \pi_P^* F \rightarrow \cdots \rightarrow \wedge^2 Q^\vee \otimes \pi_P^* F \rightarrow Q^\vee \otimes \pi_P^* F \rightarrow \pi_P^* F \rightarrow 0$$

provides a resolution of  $i_{\infty*} F$  on  $P$ . This complex is called the Koszul resolution of  $i_{\infty*} F$  and will be denoted by  $K(F, N_{X/Y})$ . If the normal bundle  $N_{X/Y}$  admits some hermitian metric, then the tautological exact sequence induces a hermitian metric on  $Q$ . If, moreover, the bundle  $F$  also admits a hermitian metric, then the Koszul resolution is a complex of hermitian vector bundles and will be denoted by  $K(\overline{F}, \overline{N}_{X/Y})$ .

We now summarize the most important result about the application of the deformation to the normal cone.

**Theorem 4.3.** Let  $i : Y \hookrightarrow X$  be a closed immersion of projective manifolds, and let  $W = W(i)$  be the deformation to the normal cone of  $i$ . Assume that  $\overline{\eta}$  is a hermitian vector bundle on  $Y$  and  $\overline{\xi}_\cdot$  is a complex of hermitian vector bundles which provides a resolution of  $i_* \overline{\eta}$  on  $X$ . Then there exists a complex of hermitian vector bundles  $\text{tr}_1(\overline{\xi}_\cdot)$  on  $W$  such that

- (i).  $\text{tr}_1(\overline{\xi}_\cdot)$  provides a resolution of  $j_* p_Y^*(\overline{\eta})$  on  $W$ ;
- (ii).  $\text{tr}_1(\overline{\xi}_\cdot)|_{X \times \{0\}}$  is isometric to the original complex  $\overline{\xi}_\cdot$ ;
- (iii). the restriction of  $\text{tr}_1(\overline{\xi}_\cdot)$  to  $\tilde{X}$  is orthogonally split;
- (iv). the restriction of  $\text{tr}_1(\overline{\xi}_\cdot)$  to  $P$  fits an exact sequence of resolutions on  $P$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overline{A}_\cdot & \longrightarrow & \text{tr}_1(\overline{\xi}_\cdot)|_P & \longrightarrow & K(\overline{\eta}, \overline{N}_{X/Y}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & i_{\infty*} \overline{\eta} & \xrightarrow{=} & i_{\infty*}(\overline{\eta}) \end{array}$$

where  $\overline{A}$  is orthogonally split and  $K(\overline{\eta}, \overline{N}_{X/Y})$  is the hermitian Koszul resolution;

(v). when  $Y = \emptyset$ ,  $\text{tr}_1(\overline{\xi})$  is the first transgression exact sequence introduced in Remark 2.3;

(vi). Let  $f : X' \rightarrow X$  be a morphism of projective manifolds which is smooth or transversal to  $Y$ . Formulate the following Cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{i} & X \end{array}$$

and denote by  $f_W$  the induced morphism from  $W' = W(i')$  to  $W$ , then we have

$$f_W^*(\text{tr}_1(\overline{\xi})) = \text{tr}_1(f^*\overline{\xi}).$$

*Proof.* If  $E$  is a vector bundle on  $X$ , we shall denote by  $E(i)$  the vector bundle on  $X \times \mathbb{P}^1$  given by  $E(i) = p_X^*E \otimes q_X^*\mathcal{O}_{\mathbb{P}^1}(i)$ . Now let  $\tilde{C}$  be the complex of vector bundles on  $X \times \mathbb{P}^1$  given by  $\tilde{C}_i = \xi_i(i) \oplus \xi_{i-1}(i-1)$  with differential  $d(a, b) = (b, 0)$ . Let  $y$  be a section of  $\mathcal{O}_{\mathbb{P}^1}(1)$  vanishing only at infinity, then on  $X \times (\mathbb{P}^1 \setminus \{\infty\})$  we may construct a family of inclusions of vector bundles  $\gamma_i : \xi_i \hookrightarrow \tilde{C}_i$  given by  $s \mapsto (s \otimes y^i, (-1)^i ds \otimes y^{i-1})$ .

On the other hand, define a complex of vector bundles  $\tilde{D}$  on  $X \times \mathbb{P}^1$  by  $\tilde{D}_i = \xi_{i-1}(i) \oplus \xi_{i-2}(i-1)$ . The morphism of complexes  $\varphi : \tilde{C} \rightarrow \tilde{D}$  given by  $\varphi(s, t) = (ds + (-1)^i t \otimes y, dt)$  induces a morphism of complexes on  $W$

$$\phi : \pi^*\tilde{C} \longrightarrow \pi^*\tilde{D}.$$

where  $\pi$  is the blow-down map. Then  $\text{tr}_1(\xi)$  is defined as the kernel of  $\phi$ .

Over  $\pi^{-1}(X \times (\mathbb{P}^1 \setminus \{\infty\}))$  we shall endow  $\text{tr}_1(\xi)_i$  with the metric induced by the identification with  $\xi_i$ . And over  $\pi^{-1}(X \times (\mathbb{P}^1 \setminus \{0\}))$  we shall endow  $\text{tr}_1(\xi)_i$  with the metric induced by  $\tilde{C}_i$ . Finally we glue together these two metrics by a partition of unity so that we get a hermitian metric on  $\text{tr}_1(\xi)$ . We refer to [2, Section 4] for the proof of the statement that the complex of hermitian vector bundles  $\text{tr}_1(\overline{\xi})$  constructed in the way above really satisfies those conditions in our theorem.  $\square$

**Remark 4.4.** (i). Assume that  $X$  is a  $\mu_n$ -equivariant projective manifold and  $E$  is an equivariant locally free sheaf on  $X$ . Then according to [13, (1.4) and (1.5)],  $\mathbb{P}(E)$  admits a canonical  $\mu_n$ -equivariant structure such that the projection map  $\mathbb{P}(E) \rightarrow X$  is equivariant and the canonical bundle  $\mathcal{O}(1)$  admits an equivariant structure. Moreover, let  $Y \rightarrow X$  be an equivariant closed immersion of projective manifolds, according to [13, (1.6)] the action of  $\mu_n$  on  $X$  can be extended to the blowing up  $\text{Bl}_Y X$  such that the blow-down map is equivariant and the canonical bundle  $\mathcal{O}(1)$  admits an equivariant structure. So the constructions of blowing up and the deformation to the normal cone are both compatible with the equivariant setting.

(ii). Furthermore, by endowing  $\mathbb{P}^1$  with the trivial action, we would like to reformulate all results in this section especially Theorem 4.3 in the equivariant setting. We first claim that the

constructions of all vector bundles and bundle morphisms in Theorem 4.3 also fit the equivariant setting. This follows from the fact that they are all constructed canonically. For more details, see [7, Exp. VII, Lemme 2.4, Proposition 2.5 and Lemme 3.2] as well as [2, Lemma 4.1, Remark (ii) p. 314 and (4.7) p. 315]. But unfortunately, the local uniqueness of resolutions (cf. [6, Theorem 8]) may not be valid for the equivariant case so that the local method used in the proof of the statement that the restriction of  $\mathrm{tr}_1(\tilde{\xi}.)$  to  $\tilde{X}$  is orthogonally split is not compatible with the equivariant setting. We have to formulate relative results and proofs in a different way.

**Lemma 4.5.** *Let  $X$  be a  $\mu_n$ -equivariant projective manifold, then the category of coherent  $\mu_n$ -modules on  $X$  is an abelian category. A complex of  $\mu_n$ -equivariant coherent sheaves on  $X$  is exact if and only if the underlying complex of  $\mathcal{O}_X$ -modules is exact.*

*Proof.* This follows from [13, Lemma 1.3]. □

**Lemma 4.6.** *Let  $X$  be a  $\mu_n$ -equivariant projective manifold. In other words,  $X$  is a projective manifold which admits an automorphism  $g$  of order  $n$ . Assume that*

$$\bar{\varepsilon} : 0 \rightarrow \bar{L} \rightarrow \bar{E} \rightarrow \bar{F} \rightarrow 0$$

*is a short exact sequence of equivariant hermitian vector bundles on  $X$ . If the underlying sequence of hermitian vector bundles is orthogonally split, then  $\bar{\varepsilon}$  is equivariantly and orthogonally split on  $X$ .*

*Proof.* Denote by  $f$  the bundle morphism  $E \rightarrow F$ , by assumption  $f$  is equivariant. Since the underlying sequence of hermitian vector bundles is orthogonally split, there exists a bundle morphism  $h$  from  $F$  to  $E$  such that  $f \circ h = \mathrm{Id}_F$  and  $\bar{F}$  is isometric to its image under this morphism  $h$ . We recall that the  $g$ -structure on  $\bar{E}$  (resp.  $\bar{F}$ ) is an isometry  $\sigma_E : g^*\bar{E} \rightarrow \bar{E}$  (resp.  $\sigma_F : g^*\bar{F} \rightarrow \bar{F}$ ) which satisfies certain associativity properties. We define a  $g$ -action on the morphisms of equivariant bundles as follows. Let  $u : M \rightarrow N$  be a morphism of equivariant bundles, then

$$g \bullet u := \sigma_N \circ g^*u \circ \sigma_M^{-1}$$

which is still a morphism from  $M$  to  $N$ . By definition,  $u$  is equivariant if and only if  $g \bullet u = u$ . One can easily check that  $g \bullet (g \bullet u) = g^2 \bullet u$ . Now since the morphisms  $f$  and  $\mathrm{Id}_F$  are both equivariant, we compute

$$\begin{aligned} \mathrm{Id}_F &= g \bullet \mathrm{Id}_F = g \bullet (f \circ h) \\ &= \sigma_F \circ g^*(f \circ h) \circ \sigma_F^{-1} = \sigma_F \circ g^*f \circ g^*h \circ \sigma_F^{-1} \\ &= \sigma_F \circ g^*f \circ \sigma_E^{-1} \circ \sigma_E \circ g^*h \circ \sigma_F^{-1} = f \circ (g \bullet h). \end{aligned}$$

Replacing  $g$  by  $g^k$  from  $k = 2$  to  $k = n$ , we get a meaningful average of  $h$  and it satisfies the following identity

$$f \circ \left( \frac{\sum_{k=0}^{n-1} g^k \bullet h}{n} \right) = \mathrm{Id}_F.$$

Therefore  $\frac{1}{n} \sum_{k=0}^{n-1} g^k \bullet h$  is an equivariant section of  $f$  which still makes  $\bar{F}$  isometric to its image, so we are done. □

**Remark 4.7.** In general, if the action on  $X$  is not of finite order or the base field of  $X$  has characteristic dividing  $n$  then the proof given for Lemma 4.6 fails. Nevertheless, we can show that  $\bar{\varepsilon}$  is always equivariantly and orthogonally split on  $X_g$ .

Actually, the problem that  $\bar{\varepsilon}$  may not be equivariantly and orthogonally split on the whole manifold  $X$  arises because  $h$  may not be equivariant. Note that on the fixed point submanifold  $X_g$ , the morphism  $h|_{(F|_{X_g})}$  is equivariant if and only if it maps  $F_\zeta$  into  $E_\zeta$  for any  $\zeta \in S^1$ . But this is rather clear because  $f$  is equivariant and the restriction of  $f \circ h$  on  $F|_{X_g}$  is exactly the identity map on  $F|_{X_g}$ . So we are done.

Together with Lemma 4.5, Lemma 4.6 and Remark 4.4, we have the following theorem which is an analogue of Theorem 4.3 in the equivariant setting.

**Theorem 4.8.** *Let  $i : Y \hookrightarrow X$  be an equivariant closed immersion of equivariant projective manifolds, and let  $W = W(i)$  be the deformation to the normal cone of  $i$ . Assume that  $\bar{\eta}$  is an equivariant hermitian vector bundle on  $Y$  and  $\bar{\xi}_\bullet$  is a complex of equivariant hermitian vector bundles which provides a resolution of  $i_*\bar{\eta}$  on  $X$ . Then there exists a complex of equivariant hermitian vector bundles  $\text{tr}_1(\bar{\xi}_\bullet)$  on  $W$  such that*

- (i).  $\text{tr}_1(\bar{\xi}_\bullet)$  provides an equivariant resolution of  $j_*p_Y^*(\bar{\eta})$  on  $W$ ;
- (ii).  $\text{tr}_1(\bar{\xi}_\bullet)|_{X \times \{0\}}$  is isometric to the original complex  $\bar{\xi}_\bullet$ ;
- (iii). the restriction of  $\text{tr}_1(\bar{\xi}_\bullet)$  to  $\tilde{X}$  is equivariantly and orthogonally split;
- (iv). the restriction of  $\text{tr}_1(\bar{\xi}_\bullet)$  to  $P$  fits an equivariant exact sequence of equivariant resolutions on  $P$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{A}_\bullet & \longrightarrow & \text{tr}_1(\bar{\xi}_\bullet)|_P & \longrightarrow & K(\bar{\eta}, \bar{N}_{X/Y}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & i_{\infty*}\bar{\eta} & \xrightarrow{=} & i_{\infty*}(\bar{\eta}) \end{array}$$

where  $\bar{A}_\bullet$  is an equivariantly and orthogonally split complex,  $K(\bar{\eta}, \bar{N}_{X/Y})$  is the hermitian Koszul resolution;

- (v). when  $Y = \emptyset$ ,  $\text{tr}_1(\bar{\xi}_\bullet)$  is the first transgression exact sequence introduced in Remark 2.3;
- (vi). Let  $f : X' \rightarrow X$  be an equivariant morphism of equivariant projective manifolds which is smooth or transversal to  $Y$ . Formulate the following Cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{i} & X \end{array}$$

and denote by  $f_W$  the induced morphism from  $W' = W(i')$  to  $W$ , then we have

$$f_W^*(\text{tr}_1(\bar{\xi}_\bullet)) = \text{tr}_1(f^*\bar{\xi}_\bullet).$$

To end this section, we recall some basic facts concerning the relation between equivariant setting and non-equivariant setting. Their proofs can be found in [14, Section 2 and 6.2].

**Proposition 4.9.** *Let  $i : Y \hookrightarrow X$  be an equivariant closed immersion of projective manifolds, and let  $i_g : Y_g \hookrightarrow X_g$  be the induced closed immersion between fixed point submanifolds. Then we have*

- (i). *the natural morphism  $N_{X_g/Y_g} \rightarrow (N_{X/Y})_g$  is an isomorphism;*
- (ii). *the natural morphism from the deformation to the normal cone  $W(i_g)$  to the fixed point submanifold  $W(i)_g$  is a closed immersion, this closed immersion induces the closed immersions  $\mathbb{P}(N_{X_g/Y_g} \oplus \mathcal{O}_{Y_g}) \rightarrow \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)_g$  and  $\widetilde{X}_g \rightarrow \widetilde{X}_g$ ;*
- (iii). *the fixed point submanifold of  $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$  is  $\mathbb{P}(N_{X_g/Y_g} \oplus \mathcal{O}_{Y_g}) \coprod_{\zeta \neq 1} \mathbb{P}((N_{X/Y})_\zeta)$ ;*
- (iv). *the closed immersion  $i_{\infty,g}$  factors through  $\mathbb{P}(N_{X_g/Y_g} \oplus \mathcal{O}_{Y_g})$  and the other components  $\mathbb{P}((N_{X/Y})_\zeta)$  don't meet  $Y$ . Hence the complex  $K(\mathcal{O}_Y, N_{X/Y})_g$ , obtained by taking the 0-degree part of the Koszul resolution, provides a resolution of  $\mathcal{O}_{Y_g}$  on  $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)_g$ .*

## 5 Equivariant singular Bott-Chern classes

Assume that  $X$  is a  $\mu_n$ -equivariant projective manifold and  $S$  is a closed conical subset of  $T_{\mathbb{R}}^*X_0$ , we fix the following notations:

$$\begin{aligned}\widetilde{\mathcal{U}}(X) &= \bigoplus_{p \geq 0} (D^{p,p}(X) / (\text{Im} \partial + \text{Im} \bar{\partial})) \\ \widetilde{\mathcal{U}}(X, S) &= \bigoplus_{p \geq 0} (D^{p,p}(X, S) / (\text{Im} \partial + \text{Im} \bar{\partial})).\end{aligned}$$

**Definition 5.1.** Let  $i : Y \hookrightarrow X$  be an equivariant closed immersion of projective manifolds. Let  $N$  be the normal bundle of this immersion and let  $h_N$  be an invariant hermitian metric on  $N$ , we shall denote  $\overline{N} = (N, h_N)$ . Moreover, let  $\overline{\eta} = (\eta, h_\eta)$  be an equivariant hermitian vector bundle on  $Y$  and let  $\overline{\xi}$  be a complex of equivariant hermitian vector bundles on  $X$  which provides a resolution of  $i_* \overline{\eta}$ . The four-tuple

$$\overline{\Xi} = (i, \overline{N}, \overline{\eta}, \overline{\xi})$$

is called an equivariant hermitian embedded vector bundle. Notice that an exact sequence of equivariant hermitian vector bundles on  $X$  is a particular case of equivariant hermitian embedded vector bundle.

**Definition 5.2.** An equivariant singular Bott-Chern class for an equivariant hermitian embedded vector bundle  $\overline{\Xi} = (i, \overline{N}, \overline{\eta}, \overline{\xi})$  is a class  $\widetilde{H} \in \widetilde{\mathcal{U}}(X_g)$  such that

$$\text{dd}^c \widetilde{H} = \sum_j (-1)^j [\text{ch}_g(\overline{\xi}_j)] - i_{g*}([\text{ch}_g(\overline{\eta}) \text{Td}_g^{-1}(\overline{N})]).$$



Note that the current

$$\sum_j (-1)^j [\text{ch}_g(\bar{\xi}_j)] - i_{g*}([\text{ch}_g(\bar{\eta}) \text{Td}_g^{-1}(\bar{N})]) = \sum_j (-1)^j [\text{ch}_g(\bar{\xi}_j)] - [\text{ch}_g(\bar{\eta}) \text{Td}_g^{-1}(\bar{N})] \delta_{Y_g}$$

is an element in  $D^*(X_g, N_{g,0}^\vee)$ , we would like to control the singularities of the Bott-Chern class so that they are contained in the same wave front set and we may do the pull-backs of currents in certain situations. Theorem 3.9 allows us to do this.

**Proposition 5.3.** *Let  $\bar{\Xi} = (i, \bar{N}, \bar{\eta}, \bar{\xi})$  be an equivariant hermitian embedded vector bundle, then any equivariant singular Bott-Chern class for  $\bar{\Xi}$  belongs to  $\tilde{\mathcal{U}}(X_g, N_{g,0}^\vee)$ .*

*Proof.* Firstly, note that Theorem 3.8 (ii) implies that the natural map from  $\tilde{\mathcal{U}}(X_g, N_{g,0}^\vee)$  to  $\tilde{\mathcal{U}}(X_g)$  is injective. Then the statement in this proposition does make sense and it follows from Theorem 3.8 (i).  $\square$

Now assume that  $f : X' \rightarrow X$  is an equivariant morphism of projective manifolds which is transversal to  $Y$ . We formulate the following Cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \downarrow h & & \downarrow f \\ Y & \xrightarrow{i} & X. \end{array}$$

Since  $h^*N$  is isomorphic to the normal bundle of the immersion  $i'$  (which implies that their restrictions to the fixed point submanifolds are also isomorphic to each other) and  $f^*\xi$  provides a resolution of  $i'_*h^*\eta$  on  $X'$ , we know that the notation  $f^*\bar{\Xi} = (i', h^*\bar{N}, h^*\bar{\eta}, f^*\bar{\xi})$  does make sense. Moreover, we conclude that  $h_g^*N_g$  is isomorphic to the normal bundle of  $i'_g$ . Then by Proposition 5.3 and Theorem 3.4, for any equivariant singular Bott-Chern class  $\bar{H}$  for  $\bar{\Xi}$ , the pull-back  $f_g^*\bar{H}$  is well-defined.

To every equivariant hermitian embedded vector bundle  $\bar{\Xi} = (i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi})$ , we may associate two new equivariant hermitian embedded vector bundles. One is  $\text{tr}_1(\bar{\Xi}) := (j : \mathbb{P}_Y^1 \rightarrow W(i), \bar{N}_{W(i)/\mathbb{P}_Y^1}, p_Y^*\bar{\eta}, \text{tr}_1(\bar{\xi}))$  concerning the construction of the deformation to the normal cone, the other one is  $\bar{\Xi}_{Kos} := (i_\infty : Y \rightarrow \mathbb{P}(N \oplus \mathcal{O}_Y), \bar{N}, \bar{\eta}, K(\bar{\eta}, \bar{N}))$  concerning the construction of the Koszul resolution.

Moreover, the direct sum of an equivariant hermitian embedded vector bundle  $\bar{\Xi} = (i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi})$  with an exact sequence  $\bar{\varepsilon}$  of equivariant hermitian vector bundles on  $X$  is defined as  $\bar{\Xi} \oplus \bar{\varepsilon} := (i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi} \oplus \bar{\varepsilon})$ .

**Definition 5.4.** Let  $\Sigma$  be a set of equivariant hermitian embedded vector bundles. We say that  $\Sigma$  satisfies the condition (Hui) if

(i). any exact sequence of equivariant hermitian vector bundles on an equivariant projective manifold belongs to  $\Sigma$  and  $\Sigma$  is closed under the operation of taking direct sum with an exact sequence of equivariant hermitian vector bundles;

(ii). for any element  $\overline{\Xi} = (i : Y \rightarrow X, \overline{N}, \overline{\eta}, \overline{\xi}) \in \Sigma$  and for every equivariant morphism  $f : X' \rightarrow X$  of projective manifolds which is transversal to  $Y$ , we have  $f^*\overline{\Xi} \in \Sigma$ .

(iii). for any element  $\overline{\Xi} = (i : Y \rightarrow X, \overline{N}, \overline{\eta}, \overline{\xi}) \in \Sigma$ , the associated equivariant hermitian embedded vector bundles  $\text{tr}_1(\overline{\Xi})$  and  $\overline{\Xi}_{Kos}$  both belong to  $\Sigma$ .

**Definition 5.5.** Let  $\Sigma$  be a set of equivariant hermitian embedded vector bundles which satisfies the condition (Hui). A theory of equivariant singular Bott-Chern classes for  $\Sigma$  is an assignment which, to each  $\overline{\Xi} = (i : Y \rightarrow X, \overline{N}, \overline{\eta}, \overline{\xi}) \in \Sigma$ , assigns a class of currents

$$T(\overline{\Xi}) \in \widetilde{\mathcal{U}}(X_g)$$

satisfying the following properties.

(i). (Differential equation) The following equality holds

$$\text{dd}^c T(\overline{\Xi}) = \sum_j (-1)^j [\text{ch}_g(\overline{\xi}_j)] - i_{g*}([\text{ch}_g(\overline{\eta}) \text{Td}_g^{-1}(\overline{N})]).$$

(ii). (Functoriality) For every equivariant morphism  $f : X' \rightarrow X$  of projective manifolds which is transversal to  $Y$ , we have

$$f_g^* T(\overline{\Xi}) = T(f^* \overline{\Xi}).$$

(iii). (Normalization) Let  $\overline{A}.$  be an equivariantly and orthogonally split exact sequence of equivariant hermitian vector bundles. Then  $T(\overline{\Xi}) = T(\overline{\Xi} \oplus \overline{A}.)$ . Moreover, if  $X = \text{Spec}(\mathbb{C})$  is one point,  $Y = \emptyset$  and  $\overline{\xi} = 0$ , then  $T(\overline{\Xi}) = 0$ .

**Remark 5.6.** (i). When  $Y = \emptyset$  and  $\overline{\xi}.$  is an exact sequence of equivariant hermitian vector bundles on  $X$ , the three properties in the definition above imply that

$$T(\overline{\Xi}) = \widetilde{\text{ch}}_g(\overline{\xi}.)$$

where  $\widetilde{\text{ch}}_g$  is the equivariant Bott-Chern secondary characteristic class associated to  $\text{ch}_g$ .

(ii). According to Definition 5.4, the properties (ii) and (iii) described in the definition above are reasonable.

Throughout the rest of this section we shall assume that  $\Sigma$  is a suitable set (big enough) of equivariant hermitian embedded vector bundles and we shall also assume the existence of a theory of equivariant singular Bott-Chern classes for  $\Sigma$ . We first show the compatibility of equivariant singular Bott-Chern classes with exact sequences and equivariant Bott-Chern secondary characteristic classes.

We fix an equivariant closed immersion  $i : Y \hookrightarrow X$  of projective manifolds. Let

$$\overline{\chi} : 0 \rightarrow \overline{\eta}_n \rightarrow \cdots \rightarrow \overline{\eta}_1 \rightarrow \overline{\eta}_0 \rightarrow 0$$

be an exact sequence of equivariant hermitian vector bundles on  $Y$ , and assume that we are given a family of equivariant hermitian embedded vector bundles  $\{\bar{\Xi}_j = (i, \bar{N}, \bar{\eta}_j, \bar{\xi}_j, \cdot)\}_{j=0}^n$  which fit the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{\xi}_{n,\cdot} & \longrightarrow & \cdots & \longrightarrow & \bar{\xi}_{1,\cdot} & \longrightarrow & \bar{\xi}_{0,\cdot} & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i_*\bar{\eta}_n & \longrightarrow & \cdots & \longrightarrow & i_*\bar{\eta}_1 & \longrightarrow & i_*\bar{\eta}_0 & \longrightarrow & 0 \end{array}$$

with exact rows. For each  $k$ , we write  $\bar{\varepsilon}_k$  for the exact sequence

$$0 \rightarrow \bar{\xi}_{n,k} \rightarrow \cdots \rightarrow \bar{\xi}_{1,k} \rightarrow \bar{\xi}_{0,k} \rightarrow 0.$$

**Proposition 5.7.** *Let notations and assumptions be as above, then we have the following equality in  $\tilde{\mathcal{U}}(X_g)$*

$$T\left(\bigoplus_{j \text{ even}} \bar{\Xi}_j\right) - T\left(\bigoplus_{j \text{ odd}} \bar{\Xi}_j\right) = \sum_k (-1)^k [\tilde{\text{ch}}_g(\bar{\varepsilon}_k)] - i_{g*}([\text{Td}_g^{-1}(\bar{N})\tilde{\text{ch}}_g(\bar{\chi})]).$$

*Proof.* According to Theorem 4.8 (v), we have the first transgression exact sequences  $\text{tr}_1(\bar{\chi})$  on  $\mathbb{P}_Y^1$  and  $\text{tr}_1(\bar{\varepsilon}_k)$  on  $\mathbb{P}_X^1$  for each  $k$ . Denote by  $l : \mathbb{P}_Y^1 \rightarrow \mathbb{P}_X^1$  the induced morphism, then there exists an exact sequence of exact sequences

$$\cdots \rightarrow \text{tr}_1(\bar{\varepsilon}_1) \rightarrow \text{tr}_1(\bar{\varepsilon}_0) \rightarrow l_*\text{tr}_1(\bar{\chi}) \rightarrow 0.$$

We fix the following notations

$$\begin{aligned} \text{tr}_1(\bar{\chi})_+ &= \bigoplus_{j \text{ even}} \text{tr}_1(\bar{\chi})_j, & \text{tr}_1(\bar{\chi})_- &= \bigoplus_{j \text{ odd}} \text{tr}_1(\bar{\chi})_j, \\ \text{tr}_1(\bar{\varepsilon}_k)_+ &= \bigoplus_{j \text{ even}} \text{tr}_1(\bar{\varepsilon}_k)_j, & \text{tr}_1(\bar{\varepsilon}_k)_- &= \bigoplus_{j \text{ odd}} \text{tr}_1(\bar{\varepsilon}_k)_j, \end{aligned}$$

then

$$\begin{aligned} \text{tr}_1(\bar{\Xi})_+ &:= (l : \mathbb{P}_Y^1 \rightarrow \mathbb{P}_X^1, p_Y^*\bar{N}, \text{tr}_1(\bar{\chi})_+, \text{tr}_1(\bar{\varepsilon}_\cdot)_+), \\ \text{tr}_1(\bar{\Xi})_- &:= (l : \mathbb{P}_Y^1 \rightarrow \mathbb{P}_X^1, p_Y^*\bar{N}, \text{tr}_1(\bar{\chi})_-, \text{tr}_1(\bar{\varepsilon}_\cdot)_-) \end{aligned}$$

are two equivariant hermitian embedded vector bundles.

By the functoriality of the first transgression exact sequences, we obtain that

$$\text{tr}_1(\bar{\Xi})_+|_{X \times \{0\}} = \bigoplus_{j \text{ even}} \text{tr}_1(\bar{\Xi}_j), \quad \text{tr}_1(\bar{\Xi})_-|_{X \times \{0\}} = \bigoplus_{j \text{ odd}} \text{tr}_1(\bar{\Xi}_j).$$

Note that for any exact sequence of equivariant hermitian vector bundles, its first transgression exact sequence is equivariantly and orthogonally split at infinity. So we have an isometry

$$\text{tr}_1(\bar{\Xi})_+|_{X \times \{\infty\}} \cong \text{tr}_1(\bar{\Xi})_-|_{X \times \{\infty\}}.$$

Since the wave front sets of the currents  $[\log |z|^2]$  and  $T(\text{tr}_1(\bar{\Xi}_\pm))$  do not intersect (cf. [2, P. 266]), by [12, Thm. 8.2.10], their products are well-defined currents. Then in  $\mathcal{U}(\mathbb{P}_{X_g}^1)$ , we have

$$\begin{aligned} 0 &= \frac{\bar{\partial}}{2\pi i} \{ \partial \log |z|^2 \cdot (T(\text{tr}_1(\bar{\Xi})_+) - T(\text{tr}_1(\bar{\Xi})_-)) \} + \frac{\partial}{2\pi i} \{ \log |z|^2 \cdot \bar{\partial} (T(\text{tr}_1(\bar{\Xi})_+) - T(\text{tr}_1(\bar{\Xi})_-)) \} \\ &= \left( \frac{\bar{\partial}\partial}{2\pi i} \log |z|^2 \right) \cdot (T(\text{tr}_1(\bar{\Xi})_+) - T(\text{tr}_1(\bar{\Xi})_-)) - \log |z|^2 \cdot \frac{\bar{\partial}\partial}{2\pi i} (T(\text{tr}_1(\bar{\Xi})_+) - T(\text{tr}_1(\bar{\Xi})_-)) \\ &= (\delta_0 - \delta_\infty) \cdot (T(\text{tr}_1(\bar{\Xi})_+) - T(\text{tr}_1(\bar{\Xi})_-)) - \log |z|^2 \cdot \sum_k (-1)^k (\text{ch}_g(\text{tr}_1(\bar{\varepsilon}_k)_+) - \text{ch}_g(\text{tr}_1(\bar{\varepsilon}_k)_-)) \\ &\quad + \log |z|^2 \cdot l_{g*} \{ \text{ch}_g(\text{tr}_1(\bar{\chi})_+) \text{Td}_g^{-1}(p_Y^* \bar{N}) - \text{ch}_g(\text{tr}_1(\bar{\chi})_-) \text{Td}_g^{-1}(p_Y^* \bar{N}) \}. \end{aligned}$$

Finally, integrating both two sides of the equality above over  $\mathbb{P}^1$  and using the first construction of equivariant Bott-Chern secondary classes, we get the identity in this proposition.  $\square$

A totally similar argument gives a proof of the following proposition.

**Proposition 5.8.** *Let  $\bar{\Xi}_0 = (i, \bar{N}_0, \bar{\eta}, \bar{\xi})$  be an equivariant hermitian embedded vector bundle with  $\bar{N}_0 = (N, h_0)$ . Assume that  $h_1$  is another invariant metric on  $N$ , we write  $\bar{N}_1 = (N, h_1)$  and  $\bar{\Xi}_1 = (i, \bar{N}_1, \bar{\eta}, \bar{\xi})$ , then we have*

$$T(\bar{\Xi}_0) - T(\bar{\Xi}_1) = -i_{g*} [\text{ch}_g(\bar{\eta}) \widetilde{\text{Td}_g^{-1}}(N, h_0, h_1)].$$

We now turn to a special case of closed immersion of equivariant projective manifolds, namely the zero section embedding discussed before Definition 4.2. Precisely speaking, let  $Y$  be an equivariant projective manifold and let  $\bar{\eta}, \bar{N}$  be two equivariant hermitian vector bundles on  $Y$ , we denote  $P = \mathbb{P}(N \oplus \mathcal{O}_Y)$ . Let  $\pi_P : P \rightarrow Y$  be the canonical projection and let  $i_\infty : Y \rightarrow P$  be the zero section embedding. As in Definition 4.2, we shall write  $K(\bar{\eta}, \bar{N})$  for the hermitian Koszul resolution. We have already know that  $\bar{\Xi}_{\text{Kos}}(\bar{\eta}, \bar{N}) = (i_\infty, \bar{N}, \bar{\eta}, K(\bar{\eta}, \bar{N}))$  is an equivariant hermitian embedded vector bundle associated to  $\bar{\Xi}$ . Sometimes we just write it as  $K(\bar{\eta}, \bar{N})$  for simplicity.

**Theorem 5.9.** *Let  $\Sigma$  be a set of equivariant hermitian embedded vector bundles which satisfies the condition (Hwi). Assume that  $T$  is a theory of equivariant singular Bott-Chern classes for  $\Sigma$ . Then the current  $(\pi_{P_g})_* T(K(\bar{\eta}, \bar{N}))$  is  $\text{dd}^c$ -closed. Moreover, the cohomology class that it represents does not depend on the metrics on  $\eta$  and  $N$  so that it determines a characteristic class  $C_T(\eta, N) \in \bigoplus_{p \geq 0} H^{p,p}(Y_g)$ .*

*Proof.* First note that the push-forwards for currents commute with differentials by definition. Then we have

$$\begin{aligned} \text{dd}^c((\pi_{P_g})_* T(K(\bar{\eta}, \bar{N}))) &= (\pi_{P_g})_* (\text{dd}^c T(K(\bar{\eta}, \bar{N}))) \\ &= (\pi_{P_g})_* \left( \sum_k (-1)^k [\text{ch}_g(\wedge^k \bar{Q}^\vee) (\pi_{P_g})^* \text{ch}_g(\bar{\eta})] - i_{\infty, g*} [\text{ch}_g(\bar{\eta}) \text{Td}_g^{-1}(\bar{N})] \right) \\ &= ((\pi_{P_g})_* [c_{\text{rk } Q_g}(\bar{Q}_g) \text{Td}_g^{-1}(\bar{Q})] - [\text{Td}_g^{-1}(\bar{N})]) \cdot [\text{ch}_g(\bar{\eta})]. \end{aligned}$$

We claim that  $(\pi_{P_g})_*[c_{\text{rk}Q_g}(\overline{Q}_g)\text{Td}_g^{-1}(\overline{Q})] = [\text{Td}_g^{-1}(\overline{N})]$  so that  $(\pi_{P_g})_*T(K(\overline{\eta}, \overline{N}))$  is dd<sup>c</sup>-closed. Actually, one first need to notice that we have the following tautological exact sequence on  $P$

$$0 \rightarrow \overline{\mathcal{O}(-1)} \rightarrow \pi_P^*(\overline{N} \oplus \overline{\mathcal{O}_Y}) \rightarrow \overline{Q} \rightarrow 0.$$

Then, by restricting to the submanifold  $P_0 = \mathbb{P}(N_g \oplus \mathcal{O}_{Y_g})$ , we get a new exact sequence

$$0 \rightarrow \overline{\mathcal{O}(-1)}|_{P_0} \rightarrow \pi_{P_g}^*(\overline{N}|_{Y_g} \oplus \overline{\mathcal{O}_{Y_g}})|_{P_0} \rightarrow \overline{Q}|_{P_0} \rightarrow 0.$$

The 0-degree part of the exact sequence above is the tautological exact sequence on  $P_0$ . Taking the non-zero degree part of this exact sequence we get an isometry  $\overline{Q}_\perp|_{P_0} \cong (\pi_{P_0})^*(\overline{N}_\perp)$ . Notice that the hermitian complex  $\wedge^\bullet \overline{Q}^\vee$  is equivariantly and orthogonally split over  $\mathbb{P}(N)$ , so the support of  $c_{\text{rk}Q_g}(\overline{Q}_g)\text{Td}_g^{-1}(\overline{Q})$  is contained in  $P_0$ . Moreover, by [5, Cor. 3.8] we know that

$$(\pi_{P_0})_*(c_{\text{rk}Q_g}(\overline{Q}_g|_{P_0})\text{Td}^{-1}(\overline{Q}_g|_{P_0})) = \text{Td}^{-1}(\overline{N}_g).$$

Then we may compute

$$\begin{aligned} (\pi_{P_g})_*(c_{\text{rk}Q_g}(\overline{Q}_g)\text{Td}_g^{-1}(\overline{Q})) &= (\pi_{P_0})_*(c_{\text{rk}Q_g}(\overline{Q}_g)\text{Td}_g^{-1}(\overline{Q})) \\ &= (\pi_{P_0})_*(c_{\text{rk}Q_g}(\overline{Q}_g|_{P_0})\text{Td}^{-1}(\overline{Q}_g|_{P_0})\text{Td}_g^{-1}(\overline{Q}_\perp|_{P_0})) \\ &= (\pi_{P_0})_*(c_{\text{rk}Q_g}(\overline{Q}_g|_{P_0})\text{Td}^{-1}(\overline{Q}_g|_{P_0})\text{Td}_g^{-1}((\pi_{P_0})^*(\overline{N}_\perp))) \\ &= \text{Td}^{-1}(\overline{N}_g)\text{Td}_g^{-1}(\overline{N}_\perp) = \text{Td}_g^{-1}(\overline{N}) \end{aligned}$$

which completes the proof of the claim.

Now let  $h_0$  and  $h_1$  (resp.  $g_0$  and  $g_1$ ) be two invariant hermitian metrics on  $N$  (resp.  $\eta$ ). We write  $\overline{N}_i = (N, h_i)$  and  $\overline{\eta}_i = (\eta, g_i)$ . We denote also by  $h_0$  and  $h_1$  the metrics induced on  $Q^\vee$ . Then by Proposition 5.7, we have

$$\begin{aligned} &(\pi_{P_g})_*(T(K(\overline{\eta}_0, \overline{N}_0)) - T(K(\overline{\eta}_1, \overline{N}_0))) \\ &= (\pi_{P_g})_*[\sum_k (-1)^k \text{ch}_g(\wedge^k \overline{Q}_0^\vee) \pi_{P_g}^* \widetilde{\text{ch}}_g(\eta, g_0, g_1)] - [\text{Td}_g^{-1}(\overline{N}_0) \widetilde{\text{ch}}_g(\eta, g_0, g_1)]. \end{aligned}$$

So using the projection formula and our claim before, we get

$$(\pi_{P_g})_*(T(K(\overline{\eta}_0, \overline{N}_0)) - T(K(\overline{\eta}_1, \overline{N}_0))) = (\pi_{P_g})_*(T(K(\overline{\eta}_1, \overline{N}_0))).$$

On the other hand, applying Proposition 5.7 and Proposition 5.8, we have

$$\begin{aligned} &(\pi_{P_g})_*(T(K(\overline{\eta}_1, \overline{N}_0)) - T(K(\overline{\eta}_1, \overline{N}_1))) \\ &= (\pi_{P_g})_*[\sum_k (-1)^k \widetilde{\text{ch}}_g(\wedge^k Q^\vee, h_0, h_1) \pi_{P_g}^* \text{ch}_g(\overline{\eta}_1)] - [\text{ch}_g(\overline{\eta}_1) \widetilde{\text{Td}}_g^{-1}(N, h_0, h_1)] \\ &= \{(\pi_{P_g})_* \sum_k (-1)^k [\widetilde{\text{ch}}_g(\wedge^k Q^\vee, h_0, h_1)] - [\widetilde{\text{Td}}_g^{-1}(N, h_0, h_1)]\} \cdot [\text{ch}_g(\overline{\eta}_1)]. \end{aligned}$$

We construct the first transgression exact sequence of  $0 \rightarrow 0 \rightarrow (N, h_1) \rightarrow (N, h_0) \rightarrow 0$  on  $\mathbb{P}_Y^1$  so that we may have an equivariant hermitian vector bundle  $(\tilde{N}, h^{\tilde{N}})$  on  $\mathbb{P}_Y^1$  such that

$$(\tilde{N}, h^{\tilde{N}})|_{Y \times \{0\}} = (N, h_1), \quad (\tilde{N}, h^{\tilde{N}})|_{Y \times \{\infty\}} = (N, h_0).$$

Now we apply the Koszul construction to the bundles  $p_Y^* \bar{\eta}_1$  and  $(\tilde{N}, h^{\tilde{N}})$  and denote by  $\pi_W$  the canonical projection from  $W := \mathbb{P}(\tilde{N} \oplus \mathcal{O}_{\mathbb{P}_Y^1})$  to  $\mathbb{P}_Y^1$ . By the universal properties of projective space bundle and fibre product,  $\mathbb{P}(\tilde{N} \oplus \mathcal{O}_{\mathbb{P}_Y^1}) = \mathbb{P}(p_Y^* N \oplus p_Y^* \mathcal{O}_Y)$  which is isomorphic to  $\mathbb{P}(N \oplus \mathcal{O}_Y) \times \mathbb{P}^1$  and the tautological quotient bundle on  $W$  is isomorphic to  $\tilde{Q}$  whose definition is similar to that of  $\tilde{N}$ . Thus we have

$$\begin{aligned} & (\pi_{P_g})_* \sum_k (-1)^k [\text{ch}_g(\wedge^k Q^\vee, h_0, h_1)] \\ &= (\pi_{P_g})_* \sum_k (-1)^k (p_{P_g})_* ([-\log |z|^2] \cdot [\text{ch}_g(\wedge^k (\tilde{Q}, h^{\tilde{Q}}))]) \\ &= (p_{Y_g})_* (\pi_{W_g})_* \sum_k (-1)^k ([-\log |z|^2] \cdot [\text{ch}_g(\wedge^k (\tilde{Q}, h^{\tilde{Q}}))]) \\ &= (p_{Y_g})_* ([-\log |z|^2] \cdot [\text{Td}_g^{-1}(\tilde{N}, h^{\tilde{N}})]) = [\widetilde{\text{Td}_g^{-1}}(N, h_0, h_1)]. \end{aligned}$$

So we have proved that  $(\pi_{P_g})_* T(K(\bar{\eta}, \bar{N}))$  dose not depend on the choices of the metrics. Thus we have a well-defined class  $C_T(\eta, N)$ . The fact that this characteristic class  $C_T(\eta, N)$  belongs to  $\bigoplus_{p \geq 0} H^{p,p}(Y_g)$  follows from [9, Theroem 1.2.2 (iii)].  $\square$

## 6 Classification of theories of equivariant singular Bott-Chern classes

The aim of this section is to give some results concerning the classification of all possible theories of equivariant singular Bott-Chern classes. We shall prove that a theory of equivariant singular Bott-Chern classes  $T$  is totally determined by the characteristic class  $C_T$  introduced in last section. Our main theorem is the following.

**Theorem 6.1.** *Let  $\Sigma$  be a set of equivariant hermitian embedded vector bundles which satisfies the condition (Hui). Assume that  $T$  and  $T'$  are two theories of equivariant singular Bott-Chern classes for  $\Sigma$ . Then  $T = T'$  if and only if for any  $(i, \bar{N}, \bar{\eta}, \bar{\xi}) \in \Sigma$ ,  $C_T(\eta, N) = C_{T'}(\eta, N)$ .*

*Proof.* One direction is clear. For the other one, we assume that  $C_T = C_{T'}$ . Let  $\bar{\Xi} = (i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi})$  be an element in  $\Sigma$ . As before, we denote by  $W$  the deformation to the normal cone and denote by  $p_W$  the composition of  $p_X$  and the blow-down map  $\pi$ . Moreover, we write  $p_{\tilde{X}} : \tilde{X} \rightarrow X$  and  $p_P : P \rightarrow X$  for the morphisms induced by  $p_W$ . The morphism  $p_P$  can be factored as  $i \circ \pi_P$ .

The normal bundle of the immersion  $j : Y \times \mathbb{P}^1 \rightarrow W$  is isomorphic to  $p_Y^* N \otimes q_Y^* \mathcal{O}(-1)$ . We endow it with the hermitian metric induced by the metric on  $N$  and the Fubini-Study metric on  $\mathcal{O}(-1)$ , the corresponding hermitian vector bundle will be denoted by  $\overline{N}'$ .

By Theorem 4.8, the restriction of  $\text{tr}_1(\overline{\xi})$  to  $X \times \{0\}$  is isometric to  $\overline{\xi}$ . and the restriction of  $\text{tr}_1(\overline{\xi})$  to  $\tilde{X}$  is equivariantly and orthogonally split. Moreover, the restriction of  $\text{tr}_1(\overline{\xi})$  to  $P$  fits an exact sequence

$$0 \rightarrow \overline{A} \rightarrow \text{tr}_1(\overline{\xi})|_{P \rightarrow K(\overline{\eta}, \overline{N})} \rightarrow 0$$

where  $\overline{A}$  is an equivariantly and orthogonally split exact sequence. We denote by  $\overline{\varepsilon}_k$  the exact sequence of the following exact sequence of equivariant hermitian vector bundles

$$0 \rightarrow \overline{A}_k \rightarrow \text{tr}_1(\overline{\xi})_k|_{P \rightarrow K(\overline{\eta}, \overline{N})_k} \rightarrow 0.$$

Next, we write  $U$  for the current  $[-\log |z|^2]$  on  $\mathbb{P}^1$  associated to a locally integrable differential form. Its pull-back to  $W_g$  is also locally integrable hence defines a current on  $W_g$  which will be also denoted by  $U$ . Note that  $q_{W(i_g)} = q_{W_g} \circ i_{W(i_g)}$  where  $i_{W(i_g)}$  is the natural immersion from  $W(i_g)$  to  $W_g$  and the wave front set of  $T(\text{tr}_1(\overline{\Xi}))$  is contained in the conormal bundle  $N_g'^\vee$ . Hence the wave front sets of  $U$  and  $T(\text{tr}_1(\overline{\Xi}))$  are disjoint so that their product  $U \cdot T(\text{tr}_1(\overline{\Xi}))$  is a well-defined current on  $W_g$ . Then, using the properties of equivariant singular Bott-Chern classes in Definition 5.5, the equality

$$\begin{aligned} 0 &= \text{dd}^c(p_{W_g})_*(U \cdot T(\text{tr}_1(\overline{\Xi}))) \\ &= (p_{\tilde{X}_g})_*(T(\text{tr}_1(\overline{\Xi}))|_{\tilde{X}_g}) + (p_{P_g})_*(T(\text{tr}_1(\overline{\Xi}))|_{P_g}) - T(\overline{\Xi}) \\ &\quad - (p_{W_g})_*(U \cdot (\sum_k (-1)^k \text{ch}_g(\text{tr}_1(\overline{\xi})_k) - j_{g*}(\text{ch}_g(p_Y^* \overline{\eta}) \text{Td}_g^{-1}(\overline{N}')))) \end{aligned}$$

holds in  $\tilde{\mathcal{U}}(X_g)$ . Notice that  $T(\text{tr}_1(\overline{\Xi}))|_{\tilde{X}_g} = T(\text{tr}_1(\overline{\Xi})|_{\tilde{X}}) = \tilde{\text{ch}}_g(\text{tr}_1(\overline{\xi})|_{\tilde{X}}) = 0$  and by Proposition 5.7, we have

$$T(\text{tr}_1(\overline{\Xi}))|_{P_g} = T(\text{tr}_1(\overline{\Xi})|_P) = T(K(\overline{\eta}, \overline{N})) - \sum_k (-1)^k [\tilde{\text{ch}}_g(\overline{\varepsilon}_k)].$$

Moreover, using the factorization of  $p_P$ , we have

$$(p_{P_g})_* T(K(\overline{\eta}, \overline{N})) = i_{g*}(\pi_{P_g})_* T(K(\overline{\eta}, \overline{N})) = i_{g*} C_T(\eta, N).$$

By the properties of the Fubini-Study metric,  $\text{ch}_g(p_Y^* \overline{\eta}) \text{Td}_g^{-1}(\overline{N}')$  is invariant under the involution on  $\mathbb{P}^1$  which sends  $z$  to  $1/z$ . Thus we obtain

$$(p_{W_g})_*(U \cdot (j_{g*}(\text{ch}_g(p_Y^* \overline{\eta}) \text{Td}_g^{-1}(\overline{N}')))) = i_{g*}(p_{Y_g})_*(U \cdot (\text{ch}_g(p_Y^* \overline{\eta}) \text{Td}_g^{-1}(\overline{N}')))) = 0$$

since the current  $U$  really changes its sign under the involution  $z \rightarrow 1/z$ . Gathering all computations above we finally get the following current equation

$$T(\overline{\Xi}) = -(p_{W_g})_*(U \cdot \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\overline{\xi})_k)) - \sum_k (-1)^k (p_{P_g})_* [\tilde{\text{ch}}_g(\overline{\varepsilon}_k)] + i_{g*} C_T(\eta, N).$$

A similar current equation for  $T'$  can be obtained in the same way. By our assumption we have  $C_T(\eta, N) = C_{T'}(\eta, N)$ , so that  $T(\bar{\Xi}) = T'(\bar{\Xi})$ . This completes the whole proof.  $\square$

From the proof of Theorem 6.1, it is natural to guess that if we are given an explicit definition of the equivariant characteristic class  $C$ , we then get a theory of equivariant singular Bott-Chern classes  $T$  such that  $C_T$  is exactly  $C$ . We prove this conjecture in the following theorem.

**Theorem 6.2.** *Let  $\Sigma$  be a set of equivariant hermitian embedded vector bundles which satisfies the condition (Hui). Assume that  $C$  is an equivariant characteristic class for pairs of vector bundles  $(\eta, N)$  which appear in the elements  $(i, \bar{N}, \bar{\eta}, \bar{\xi}) \in \Sigma$ . Then there exists a theory of equivariant singular Bott-Chern classes for  $\Sigma$  such that  $C_T = C$ .*

*Proof.* For any element  $\bar{\Xi} = (i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi}) \in \Sigma$ , we define

$$T(\bar{\Xi}) = -(p_{W_g})_*(U \cdot \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\bar{\xi})_k)) - \sum_k (-1)^k (p_{P_g})_*[\tilde{\text{ch}}_g(\bar{\varepsilon}_k)] + i_{g*} C(\eta, N).$$

Our first aim is to prove that such  $T$  does not depend on the choice of the metric on  $\text{tr}_1(\xi)_k$  or on  $A$ . and that such  $T$  satisfies all properties in the definition of a theory of equivariant singular Bott-Chern classes.

We denote by  $h_k$  and  $h'_k$  (resp.  $g_k$  and  $g'_k$ ) two invariant hermitian metrics on  $\text{tr}_1(\xi)_k$  (resp.  $A_k$ ) such that the resulting hermitian vector bundles all satisfy the requirements in Theorem 4.8. Then, in  $\tilde{\mathcal{U}}(X_g)$ , we have

$$\begin{aligned} & \sum_k (-1)^k (p_{P_g})_*[\tilde{\text{ch}}_g(\bar{\varepsilon}_k)] - \sum_k (-1)^k (p_{P_g})_*[\tilde{\text{ch}}_g(\bar{\varepsilon}'_k)] \\ &= \sum_k (-1)^k (p_{P_g})_*[\tilde{\text{ch}}_g(A_k, g_k, g'_k)] - \sum_k (-1)^k (p_{P_g})_*[\tilde{\text{ch}}_g(\text{tr}_1(\xi)_k |_{P, h_k, h'_k})]. \end{aligned}$$

The first term of the right-hand side vanishes due to Proposition 5.7 and the assumption that the complex  $A$  is orthogonally split for both metrics.

On the other hand, we have by definition

$$\begin{aligned} & (p_{W_g})_*(U \cdot \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\xi)_k, h_k)) - (p_{W_g})_*(U \cdot \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\xi)_k, h'_k)) \\ &= (p_{W_g})_*(U \cdot \sum_k (-1)^k \text{dd}^c \tilde{\text{ch}}_g(\text{tr}_1(\xi)_k, h_k, h'_k)). \end{aligned}$$

But, in  $\tilde{\mathcal{U}}(X_g)$ , we have

$$\begin{aligned} & (p_{W_g})_*(U \cdot \sum_k (-1)^k \text{dd}^c \tilde{\text{ch}}_g(\text{tr}_1(\xi)_k, h_k, h'_k)) \\ &= \sum_k (-1)^k (p_{\tilde{X}_g})_*[\tilde{\text{ch}}_g(\text{tr}_1(\xi)_k, h_k, h'_k)] |_{\tilde{X}_g} + \sum_k (-1)^k (p_{P_g})_*[\tilde{\text{ch}}_g(\text{tr}_1(\xi)_k, h_k, h'_k)] |_{P_g} \\ & \quad - \sum_k (-1)^k [\tilde{\text{ch}}_g(\text{tr}_1(\xi)_k, h_k, h'_k)] |_{X \times \{0\}}. \end{aligned}$$



The last term of the right-hand side vanishes because the metrics  $h_k$  and  $h'_k$  agree each other on  $X \times \{0\}$ . The first term vanishes due to the assumption that  $\text{tr}_1(\xi.)|_{\tilde{X}}$  is orthogonally split with both metrics. Therefore, combining the two computations above, we know that the definition of  $T$  is independent of the metrics on  $A$ . and  $\text{tr}_1(\xi.)$ .

We next prove that the definition of  $T$  satisfies the three properties in the definition of a theory of equivariant singular Bott-Chern classes. For the differential equation, we compute

$$\begin{aligned} \text{dd}^c T(\Xi) &= - \sum_k (-1)^k (p_{\tilde{X}g})_* \text{ch}_g(\text{tr}_1(\bar{\xi}.)_k |_{\tilde{X}}) - \sum_k (-1)^k (p_{Pg})_* \text{ch}_g(\text{tr}_1(\bar{\xi}.)_k |_P) \\ &\quad + \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\bar{\xi}.)_k |_{X \times \{0\}}) \\ &\quad - \sum_k (-1)^k (p_{Pg})_* (\text{ch}_g(\bar{A}_k) + \text{ch}_g(K(\bar{\eta}, \bar{N})_k) - \text{ch}_g(\text{tr}_1(\bar{\xi}.)_k |_P)). \end{aligned}$$

Using the fact that  $\bar{A}$ . and  $\text{tr}_1(\bar{\xi}.)|_{\tilde{X}}$  are equivariantly and orthogonally split we obtain

$$\begin{aligned} \text{dd}^c T(\Xi) &= \sum_k (-1)^k \text{ch}_g(\bar{\xi}_k) - \sum_k (-1)^k (p_{Pg})_* \text{ch}_g(K(\bar{\eta}, \bar{N})_k) \\ &= \sum_k (-1)^k [\text{ch}_g(\bar{\xi}_k)] - (p_{Pg})_* [c_{\text{rk} Q_g}(\bar{Q}_g) \text{Td}_g^{-1}(\bar{Q}) \text{ch}_g(\pi_P^* \bar{\eta})] \\ &= \sum_k (-1)^k [\text{ch}_g(\bar{\xi}_k)] - i_{g*} [\text{ch}_g(\bar{\eta}) \text{Td}_g^{-1}(\bar{N})]. \end{aligned}$$

Secondly, the functoriality property for our definition of  $T$  follows from the functoriality property for  $\text{ch}_g$ ,  $\tilde{\text{ch}}_g$  and  $C$ .

We now prove the normalization property. We first assume that  $Y = \emptyset$  and  $\bar{\xi}.$  is an equivariantly and orthogonally split exact sequence. This means that if we write  $\bar{K}_i = \text{Ker}(d_i : \bar{\xi}_i \rightarrow \bar{\xi}_{i-1})$ , then  $\bar{\xi}_i$  is isometric to  $\bar{K}_i \oplus \bar{K}_{i-1}$ . Hence by the construction of  $\text{tr}_1(\xi.)$ , we know that

$$\text{tr}_1(\xi.)_i = p_X^* K_i \otimes q_X^* \mathcal{O}(i) \oplus p_X^* K_{i-1} \otimes q_X^* \mathcal{O}(i-1).$$

This formula implies that  $\sum_k (-1)^k \text{ch}_g(\text{tr}_1(\bar{\xi}.)_k)$  is invariant under the involution on  $\mathbb{P}^1$  which sends  $z$  to  $1/z$ . So the first term in the definition for  $T$  vanishes. It is clear that the other two terms also vanish in this special case. Hence we obtain  $T(\bar{\xi}.) = 0$ . Now let  $\Xi = (i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi}.)$  and let  $\bar{B}.$  be an equivariantly and orthogonally split exact sequence of equivariant hermitian vector bundles on  $X$ . By [10, Section 1.1], we have

$$\text{tr}_1(\xi. \oplus B.) = \text{tr}_1(\xi.) \oplus \pi^* \text{tr}_1(B.).$$

In order to compute  $T(\Xi \oplus \bar{B}.)$ , we consider the following exact sequences

$$\varepsilon'_k : 0 \rightarrow \bar{A}_k \oplus \pi^* \text{tr}_1(\bar{B}.)_k |_P \rightarrow \text{tr}_1(\bar{\xi}.)_k \oplus \pi^* \text{tr}_1(\bar{B}.)_k |_P \rightarrow K(\bar{\eta}, \bar{N})_k \rightarrow 0.$$

By the additivity of equivariant Bott-Chern secondary characteristic classes, we have  $\widetilde{\text{ch}}_g(\bar{\varepsilon}_k) = \widetilde{\text{ch}}_g(\bar{\varepsilon}'_k)$ . Again using the additivity of equivariant Chern classes, we finally get

$$T(\bar{\Xi} \oplus \bar{B}.) - T(\bar{B}.) = 0.$$

At last, we should prove that the equivariant characteristic class  $C_T$  is exactly equal to  $C$ . Note that the arguments above show that  $T$  is really a theory of equivariant singular Bott-Chern classes, then as what we have seen in the proof of Theorem 6.1, for any element  $\bar{\Xi} = (i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi}.) \in \Sigma$  we always have

$$T(\bar{\Xi}) = -(p_{W_g})_*(U \cdot \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\bar{\xi}.)_k)) - \sum_k (-1)^k (p_{P_g})_*[\widetilde{\text{ch}}_g(\bar{\varepsilon}_k)] + i_{g*} C_T(\eta, N).$$

In particular, for the Koszul construction  $(i : Y \rightarrow \mathbb{P}(N \oplus \mathcal{O}_Y), \bar{N}, \bar{\eta}, K(\bar{\eta}, \bar{N}))$ , we have

$$T(K(\bar{\eta}, \bar{N})) = -(p_{W_g})_*(U \cdot \sum_k (-1)^k \text{ch}_g(\text{tr}_1(K(\bar{\eta}, \bar{N})_k)) - \sum_k (-1)^k (p_{P_g})_*[\widetilde{\text{ch}}_g(\bar{\varepsilon}_k)] + i_{g*} C_T(\eta, N).$$

Comparing with the definition of  $T$  via the characteristic class  $C$ , we get  $i_{g*} C_T(\eta, N) = i_{g*} C(\eta, N)$  and hence  $C_T(\eta, N) = C(\eta, N)$  after composing  $(\pi_{P_g})_*$ . This completes the whole proof.  $\square$

To end this section, we shall give an example of the set of equivariant hermitian embedded vector bundles which satisfies the condition (Hui) and we shall also give a general way to construct the characteristic class  $C$ .

**Definition 6.3.** Let  $\bar{\Xi} = (i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi}.)$  be an equivariant hermitian embedded vector bundle. The equivariant rank of  $\bar{\Xi}$  is the sequence of locally constant functions  $(\text{rk} \eta_\zeta)_{\zeta \in S^1}$ . The equivariant codimension of  $\bar{\Xi}$  is the sequence of locally constant functions  $(\text{rk} N_\zeta)_{\zeta \in S^1}$ . When  $Y = \emptyset$ , we shall say that an exact sequence of equivariant hermitian vector bundles on  $X$  has arbitrary equivariant rank and arbitrary equivariant codimension.

**Proposition 6.4.** Let  $(t_\zeta)_{\zeta \in S^1}$  and  $(s_\zeta)_{\zeta \in S^1}$  be two sequences of natural numbers. Let  $\Sigma$  be a set consisting of all equivariant hermitian embedded vector bundles of equivariant rank less than or equal to  $(t_\zeta)_{\zeta \in S^1}$  and of equivariant codimension less than or equal to  $(s_\zeta)_{\zeta \in S^1}$ . Then  $\Sigma$  satisfies the condition (Hui).

*Proof.* The first requirement in the condition (Hui) is naturally fulfilled by definition. For the second requirement, let  $(i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi}.)$  be an element in  $\Sigma$  and let  $f : X' \rightarrow X$  be an equivariant morphism which is transversal to  $Y$ . Then  $f^{-1}(Y)$  either is empty set or has the same codimension as  $Y$ . In the first case, we are done. In the second case, we formulate the following Cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \downarrow h & & \downarrow f \\ Y & \xrightarrow{i} & X, \end{array}$$

then  $h^*N \cong N'$ . Note that  $(h^*\eta)|_{Y'_g} = h_g^*(\eta|_{Y_g})$  and  $(h^*N)|_{Y'_g} = h_g^*(N|_{Y_g})$ , we have the inequalities  $(\text{rk}(h^*\eta)_\zeta)_{\zeta \in S^1} \leq (t_\zeta)_{\zeta \in S^1}$  and  $(\text{rk}N'_\zeta)_{\zeta \in S^1} \leq (s_\zeta)_{\zeta \in S^1}$ . This means that the equivariant hermitian embedded vector bundle  $(i' : Y' \rightarrow X', \overline{N}', h^*\overline{\eta}, f^*\overline{\xi})$  is also in  $\Sigma$ . For the last requirement in the condition (Hui), we again let  $\overline{\Xi} = (i : Y \rightarrow X, \overline{N}, \overline{\eta}, \overline{\xi})$  be an element in  $\Sigma$ . Then the associated Koszul construction  $\overline{\Xi}_{Kos}$  clearly has the same equivariant rank and equivariant codimension as  $\overline{\Xi}$ . Concerning the construction of the deformation to the normal cone,  $\text{tr}_1(\overline{\Xi})$  clearly has the same equivariant rank as  $\overline{\Xi}$ . Moreover, the normal bundle of  $Y \times \mathbb{P}^1$  in  $W(i)$  is  $N' = p_Y^*N \otimes q_Y^*\mathcal{O}(-1)$ . Note that we assume that  $\mathbb{P}^1$  admits the trivial  $g$ -action, then  $(\text{rk}N'_\zeta)_{\zeta \in S^1} = (\text{rk}N_\zeta)_{\zeta \in S^1}$  so that  $\text{tr}_1(\overline{\Xi})$  also has the same equivariant codimension as  $\overline{\Xi}$ . Therefore, we have that  $\text{tr}_1(\overline{\Xi})$  and  $\overline{\Xi}_{Kos}$  are both elements in  $\Sigma$ .  $\square$

We finally give a general construction of the characteristic class  $C$  for the set  $\Sigma$  in last proposition.

**Definition 6.5.** Let  $(\varphi_\zeta)_{\zeta \in S^1}$  be a family of  $\mathbf{GL}(\mathbb{C})$ -invariant formal power series such that  $\varphi_\zeta \in \mathbb{C}[[\mathbf{g}_{\text{rk}\eta_\zeta}(\mathbb{C})]]$ . And let  $(\psi_\zeta)_{\zeta \in S^1}$  be a family of  $\mathbf{GL}(\mathbb{C})$ -invariant formal power series such that  $\psi_\zeta \in \mathbb{C}[[\mathbf{g}_{\text{rk}N_\zeta}(\mathbb{C})]]$ . Moreover, let  $\phi \in \mathbb{C}[[\bigoplus_{\zeta \in S^1} \mathbb{C} \oplus \bigoplus_{\zeta \in S^1} \mathbb{C}]]$  be any formal power series. We define the equivariant character form  $\phi_g(\overline{\eta}, \overline{N})$  as

$$\phi_g(\overline{\eta}, \overline{N}) = \phi((\varphi_\zeta(-\frac{\Omega \overline{\eta}_\zeta}{2\pi i}))_{\zeta \in S^1}, (\psi_\zeta(-\frac{\Omega \overline{N}_\zeta}{2\pi i}))_{\zeta \in S^1}).$$

The cohomology class that  $\phi_g(\overline{\eta}, \overline{N})$  represents is independent of the choices of the metrics, hence it define a characteristic class. We denote it by  $C(\eta, N)$ .

Then the following corollary follows immediately from Theorem 6.1 and Theorem 6.2.

**Corollary 6.6.** *Let  $\Sigma$  be the set of equivariant hermitian embedded vector bundles defined in Proposition 6.4. Let  $C$  be an equivariant characteristic class for pairs of equivariant hermitian vector bundles given in the way as in Definition 6.5. Then there exists a unique theory of equivariant singular Bott-Chern classes  $T$  for  $\Sigma$  such that  $C_T$  is equal to  $C$ .*

## 7 Compatibility with the projection formula

As usual, let  $\Sigma$  be a set of equivariant hermitian embedded vector bundles which satisfies the condition (Hui). In this section, we shall give the sufficient and necessary condition for a theory of equivariant singular Bott-Chern classes to be compatible with the projection formula. This can be regarded as an example of how the properties of the characteristic class  $C_T$  are reflected in the corresponding theory of equivariant singular Bott-Chern classes.

Now, let  $\overline{\Xi} = (i : Y \rightarrow X, \overline{N}, \overline{\eta}, \overline{\xi})$  be an equivariant hermitian embedded vector bundle in  $\Sigma$ . For any equivariant hermitian vector bundle  $\overline{\kappa}$  on  $X$ , we define

$$\overline{\Xi} \otimes \overline{\kappa} = (i : Y \rightarrow X, \overline{N}, \overline{\eta} \otimes i^*\overline{\kappa}, \overline{\xi} \otimes \overline{\kappa}).$$

Note that  $\overline{\Xi} \otimes \overline{\kappa}$  is also an equivariant hermitian embedded vector bundle according to the projection formula. We assume that  $\Sigma$  is big enough so that  $\overline{\Xi} \otimes \overline{\kappa}$  and all equivariant hermitian embedded vector bundle appearing below belong to it.

**Definition 7.1.** Let notations and assumptions be as above. A theory of equivariant singular Bott-Chern classes  $T$  for  $\Sigma$  is said to be compatible with the projection formula if

$$T(\overline{\Xi} \otimes \overline{\kappa}) = T(\overline{\Xi}) \cdot \text{ch}_g(\overline{\kappa}).$$

**Proposition 7.2.** *Let notations and assumptions be as above. Then*

$$T(\overline{\Xi} \otimes \overline{\kappa}) - T(\overline{\Xi}) \cdot \text{ch}_g(\overline{\kappa}) = i_{g*}(C_T(\eta \otimes i^* \kappa, N)) - i_{g*}(C_T(\eta, N)) \cdot \text{ch}_g(\overline{\kappa}).$$

*Proof.* As before, denote by  $p_W$  the composition of the blow-down map  $\pi$  and the projection  $p_X : X \times \mathbb{P}^1 \rightarrow X$ . Then by the construction of  $\text{tr}_1(\cdot)$ , we have  $\text{tr}_1(\overline{\xi} \otimes \overline{\kappa}) = \text{tr}_1(\overline{\xi}) \otimes p_W^* \overline{\kappa}$ . Then, on one hand, we have

$$\begin{aligned} (p_{Wg})_*(U \cdot \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\overline{\xi} \otimes \overline{\kappa})_k)) &= (p_{Wg})_*(U \cdot \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\overline{\xi})_k) p_{Wg}^* \text{ch}_g(\overline{\kappa})) \\ &= (p_{Wg})_*(U \cdot \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\overline{\xi})_k)) \text{ch}_g(\overline{\kappa}). \end{aligned}$$

On the other hand, the Koszul resolution of  $i_*(\eta \otimes i^* \kappa)$  is given by

$$K(\eta \otimes i^* \kappa, N) = K(\eta, N) \otimes p_P^* \kappa.$$

Then for each  $k$ , if we write  $\overline{\varepsilon}_k \otimes p_P^* \overline{\kappa}$  for the exact sequence

$$0 \rightarrow \overline{A}_k \otimes p_P^* \overline{\kappa} \rightarrow \text{tr}_1(\overline{\xi} \otimes \overline{\kappa})_k \xrightarrow{|P \rightarrow} K(\overline{\eta}, \overline{N})_k \otimes p_P^* \overline{\kappa} \rightarrow 0,$$

we will get

$$(p_{Pg})_*[\widetilde{\text{ch}}_g(\overline{\varepsilon}_k \otimes p_P^* \overline{\kappa})] = (p_{Pg})_*[\widetilde{\text{ch}}_g(\overline{\varepsilon}_k)(p_{Pg})^* \text{ch}_g(\overline{\kappa})] = (p_{Pg})_*[\widetilde{\text{ch}}_g(\overline{\varepsilon}_k)] \cdot [\text{ch}_g(\overline{\kappa})].$$

Combing the two computations above and the unique expression of  $T$  via  $C_T$ , we get the equality in the statement of this proposition.  $\square$

**Definition 7.3.** An equivariant characteristic class  $C$  for pairs of equivariant hermitian vector bundles is said to be compatible with the projection formula if it satisfies

$$C(\eta, N) = C(\mathcal{O}_Y, N) \cdot \text{ch}_g(\eta).$$

The following is the main theorem in this section.

**Theorem 7.4.** *A theory of equivariant singular Bott-Chern classes  $T$  for  $\Sigma$  is compatible with the projection formula if and only if the associated characteristic class  $C_T$  is so.*

*Proof.* We first assume that  $C_T$  is compatible with the projection formula, then we compute

$$\begin{aligned} i_{g*} C_T(\eta \otimes i^* \kappa, N) &= i_{g*} (C_T(\mathcal{O}_Y, N) \cdot \text{ch}_g(\eta \otimes i^* \kappa)) = i_{g*} (C_T(\mathcal{O}_Y, N) \cdot \text{ch}_g(\eta) \cdot i_g^* \text{ch}_g(\kappa)) \\ &= i_{g*} (C_T(\mathcal{O}_Y, N) \cdot \text{ch}_g(\eta)) \cdot \text{ch}_g(\kappa) = i_{g*} (C_T(\eta, N)) \cdot \text{ch}_g(\kappa). \end{aligned}$$

Therefore, by Proposition 7.2,  $T$  is compatible with the projection formula.

For the other direction, assume that  $T$  is compatible with the projection formula. Using the definition of  $C_T$ , we compute

$$\begin{aligned} C_T(\eta, N) &= (\pi_{Pg})_*(T(K(\bar{\eta}, \bar{N}))) = (\pi_{Pg})_*(T(K(\bar{\mathcal{O}}_Y, \bar{N}) \otimes \pi_P^* \bar{\eta})) \\ &= (\pi_{Pg})_*(T(K(\bar{\mathcal{O}}_Y, \bar{N})) \cdot \pi_P^* \text{ch}_g(\bar{\eta})) = (\pi_{Pg})_*(T(K(\bar{\mathcal{O}}_Y, \bar{N}))) \cdot \text{ch}_g(\bar{\eta}) \\ &= C_T(\mathcal{O}_Y, N) \cdot \text{ch}_g(\eta). \end{aligned}$$

This implies that  $C_T$  is compatible with the projection formula.  $\square$

## 8 Uniqueness of equivariant singular Bott-Chern classes

Let  $\Sigma$  be any set of equivariant hermitian embedded vector bundles which satisfies the condition (Hui) and whose elements have bounded equivariant ranks and bounded equivariant codimensions. By Corollary 6.6, it is possible to attach  $\Sigma$  a theory of equivariant singular Bott-Chern classes. But unfortunately, such a theory is not unique. Our aim in this section is to show that if we add another axiom to Definition 5.5, we will get a unique theory of equivariant singular Bott-Chern classes for  $\Sigma$  without the limitation of the bounds of equivariant rank and codimension. Such a theory will be called a theory of equivariant homogeneous singular Bott-Chern classes. We shall also compare it with the theory of equivariant singular Bott-Chern currents defined by J.-M. Bismut in [4].

Our startint point is again the Koszul construction. Let  $Y$  be an equivariant projective manifold. Assume that we are given two equivariant hermitian vector bundles  $\bar{\eta}$  and  $\bar{N}$  on  $Y$ . Let  $P = \mathbb{P}(N \oplus \mathcal{O}_Y)$ ,  $P_0 = \mathbb{P}(N_g \oplus \mathcal{O}_{Y_g})$  and let  $i_\infty$  be the zero section embedding. Suppose that  $T$  is a theory of equivariant singular Bott-Chern classes for  $\Sigma$ . Then by definition, we have

$$\text{dd}^c T(K(\bar{\eta}, \bar{N})) = c_{\text{rk} Q_g}(\bar{Q}_g |_{P_0}) \text{Td}_g^{-1}(\bar{Q}) \text{ch}_g(\pi_P^* \bar{\eta}) - (i_{\infty, g})_*(\text{ch}_g(\bar{\eta}) \text{Td}_g^{-1}(\bar{N})).$$

Therefore, the class

$$\tilde{e}_T(\bar{\eta}, \bar{N}) := T(K(\bar{\eta}, \bar{N})) \cdot \text{Td}_g(\bar{Q}) \cdot \text{ch}_g^{-1}(\pi_P^* \bar{\eta})$$

satisfies the following differential equation

$$\text{dd}^c \tilde{e}_T(\bar{\eta}, \bar{N}) = c_{\text{rk} Q_g}(\bar{Q}_g |_{P_0}) - \delta_{Y_g}.$$

Note that by our descriptions in Proposition 4.9 (iii) and (iv), the current  $c_{\text{rk} Q_g}(\bar{Q}_g |_{P_0}) - \delta_{Y_g}$  belongs to  $\text{Im}(D^{\text{rk} Q_g, \text{rk} Q_g}(P_0) \hookrightarrow D^{\text{rk} Q_g, \text{rk} Q_g}(P_g))$ . Then it is natural to introduce the following definition.

**Definition 8.1.** Let  $T$  be a theory of equivariant singular Bott-Chern classes for some  $\Sigma$ . The class  $\tilde{e}_T(\bar{\eta}, \bar{N})$  is called the Euler-Green class associated to  $T$ . We say that  $T$  is homogeneous if

$$\tilde{e}_T(\bar{\eta}, \bar{N}) \in \tilde{\mathcal{U}}^{\mathrm{rk}Q_g-1, \mathrm{rk}Q_g-1}(P_0) := D^{\mathrm{rk}Q_g-1, \mathrm{rk}Q_g-1}(P_0)/(\mathrm{Im}\partial + \mathrm{Im}\bar{\partial})$$

for any element  $(i : Y \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi}) \in \Sigma$ .

**Remark 8.2.** If  $T$  is compatible with the projection formula, then the Euler-Green class  $\tilde{e}_T(\bar{\eta}, \bar{N})$  has nothing to do with the first variable.

**Theorem 8.3.** Let  $\Sigma$  be any set of equivariant hermitian embedded vector bundles which satisfies the condition (Hui). Then there exists a unique theory of equivariant homogeneous singular Bott-Chern classes for  $\Sigma$ .

*Proof.* We shall use a uniqueness theorem of the Euler-Green class in non-equivariant case. That's the following.

**Lemma 8.4.** Let  $i_\infty : Y \rightarrow P = \mathbb{P}(N \oplus \mathcal{O}_Y)$  be a zero section embedding in non-equivariant setting. Denote  $D_\infty = \mathbb{P}(N)$ . Then there exists a unique class  $\tilde{e}(P, \bar{Q}, i_\infty) \in \tilde{\mathcal{U}}^{\mathrm{rk}Q-1, \mathrm{rk}Q-1}(P)$  such that

- (i).  $\mathrm{dd}^c \tilde{e}(P, \bar{Q}, i_\infty) = c_{\mathrm{rk}Q}(\bar{Q}) - \delta_Y$ ;
- (ii).  $\tilde{e}(P, \bar{Q}, i_\infty) |_{D_\infty} = 0$ .

We refer to [5, Lemma 9.4] for a proof of this lemma. One just need to pay attention to two points. Firstly, the restriction isomorphism on analytic Deligne cohomology should be changed to  $H^{\mathrm{rk}Q-1, \mathrm{rk}Q-1}(P) \cong H^{\mathrm{rk}Q-1, \mathrm{rk}Q-1}(D_\infty)$  on Dolbeault cohomology which can be deduced from the classical projective bundle theorem for deRham cohomology and Hodge decomposition. Secondly, the existence of a preimage of  $c_{\mathrm{rk}Q}(\bar{Q}) - \delta_Y$  under  $\mathrm{dd}^c$  is a consequence of [9, Theorem 1.2.1]. What we want to indicate is that this lemma naturally leads to a similar result in the equivariant setting by using Proposition 4.9 (iii) and (iv). The result reads: there exists a unique class  $\tilde{e}(P, \bar{Q}, i_\infty) \in \mathrm{Im}(\tilde{\mathcal{U}}^{\mathrm{rk}Q_g-1, \mathrm{rk}Q_g-1}(P_0) \hookrightarrow \tilde{\mathcal{U}}^{\mathrm{rk}Q_g-1, \mathrm{rk}Q_g-1}(P_g))$  such that  $\mathrm{dd}^c \tilde{e}(P, \bar{Q}, i_\infty) = c_{\mathrm{rk}Q_g}(\bar{Q}_g |_{P_0}) - \delta_{Y_g}$  and  $\tilde{e}(P, \bar{Q}, i_\infty) |_{D_{\infty, g}} = 0$ . Moreover, by convention, we shall identify  $D^{-1, -1}(P_g)$  with the zero space.

Now assume that  $T$  is a theory of equivariant homogeneous singular Bott-Chern classes. Since the restriction of the Koszul resolution  $K(\bar{\eta}, \bar{N})$  to  $D_{\infty, g}$  is equivariantly and orthogonally split, then we have  $T(K(\bar{\eta}, \bar{N})) |_{D_{\infty, g}} = 0$ . Thus the restriction of the Euler-Green class  $\tilde{e}_T(\bar{\eta}, \bar{N})$  to  $D_{\infty, g}$  is equal to 0 by definition. Therefore, by uniqueness, we get  $\tilde{e}_T(\bar{\eta}, \bar{N}) = \tilde{e}(P, \bar{Q}, i_\infty)$  and hence

$$T(K(\bar{\eta}, \bar{N})) = \tilde{e}(P, \bar{Q}, i_\infty) \cdot \mathrm{Td}_g^{-1}(\bar{Q}) \cdot \mathrm{ch}_g(\pi_P^* \bar{\eta}).$$

This equation implies that the characteristic class

$$C_T(\eta, N) = (\pi_{P_g})_* T(K(\bar{\eta}, \bar{N}))$$

is independent of the theory  $T$ . So the uniqueness of  $T$  follows from Theorem 6.1.

For the existence, we define

$$C(\overline{\eta}, \overline{N}) = (\pi_{P_g})_*(\tilde{e}(P, \overline{Q}, i_\infty) \cdot \mathrm{Td}_g^{-1}(\overline{Q}) \cdot \mathrm{ch}_g(\pi_P^* \overline{\eta})).$$

By the differential equation that  $\tilde{e}(P, \overline{Q}, i_\infty)$  satisfies, one can easily prove that  $C(\overline{\eta}, \overline{N})$  is  $\mathrm{dd}^c$ -closed. This is the only important point for us to use the same principle as in the proof of Theorem 6.2 to define a theory of equivariant homogeneous singular Bott-Chern classes  $T$  such that  $C_T = C$ . The last equality says that  $C(\overline{\eta}, \overline{N})$  is actually independent of the choices of the metrics since  $C_T$  is so. Thus we can just write

$$C(\eta, N) = (\pi_{P_g})_*(\tilde{e}(P, \overline{Q}, i_\infty) \cdot \mathrm{Td}_g^{-1}(\overline{Q}) \cdot \mathrm{ch}_g(\pi_P^* \overline{\eta})).$$

And this is compatible with Theorem 6.2.  $\square$

Since the class  $\tilde{e}(P, \overline{Q}, i_\infty)$  has nothing to do with the vector bundle  $\eta$ , the following remark looks more natural.

**Remark 8.5.** If  $T$  is compatible with the projection formula, then  $T$  is homogeneous if and only if  $\tilde{e}_T(\overline{\mathcal{O}_Y}, \overline{N}) = \tilde{e}(P, \overline{Q}, i_\infty)$ .

We reformulate Theorem 8.3 in an axiomatic way.

**Theorem 8.6.** *There exists a unique way to associate to each equivariant hermitian embedded vector bundle  $\overline{\Xi} = (i : Y \rightarrow X, \overline{N}, \overline{\eta}, \overline{\xi}, \cdot)$  a class of currents*

$$T^h(\overline{\Xi}) \in \tilde{\mathcal{U}}(X_g, N_{g,0}^\vee)$$

*which we call equivariant homogeneous singular Bott-Chern class, satisfying the following properties*

(i). (Differential equation) *The following equality holds*

$$\mathrm{dd}^c T^h(\overline{\Xi}) = \sum_j (-1)^j [\mathrm{ch}_g(\overline{\xi}_j)] - i_{g*}([\mathrm{ch}_g(\overline{\eta}) \mathrm{Td}_g^{-1}(\overline{N})]).$$

(ii). (Functoriality) *For every equivariant morphism  $f : X' \rightarrow X$  of projective manifolds which is transversal to  $Y$ , we have*

$$f_g^* T^h(\overline{\Xi}) = T^h(f^* \overline{\Xi}).$$

(iii). (Normalization) *Let  $\overline{A}$  be an equivariantly and orthogonally split exact sequence of equivariant hermitian vector bundles. Write  $\overline{\Xi} \oplus \overline{A} = (i, \overline{N}, \overline{\eta}, \overline{\xi} \oplus \overline{A})$ . Then  $T^h(\overline{\Xi}) = T^h(\overline{\Xi} \oplus \overline{A})$ . Moreover, if  $X = \mathrm{Spec}(\mathbb{C})$  is one point,  $Y = \emptyset$  and  $\overline{\xi} = 0$ , then  $T^h(\overline{\Xi}) = 0$ .*

(iv). (Homogeneity) *For any Koszul construction, we have*

$$T^h(K(\overline{\eta}, \overline{N})) \cdot \mathrm{Td}_g(\overline{Q}) \cdot \mathrm{ch}_g(\pi_P^* \overline{\eta}) \in \mathrm{Im}(\tilde{\mathcal{U}}^{\mathrm{rk} Q_g - 1, \mathrm{rk} Q_g - 1}(P_0) \hookrightarrow \tilde{\mathcal{U}}^{\mathrm{rk} Q_g - 1, \mathrm{rk} Q_g - 1}(P_g)).$$

**Proposition 8.7.** *The theory of equivariant homogeneous singular Bott-Chern classes is compatible with the projection formula.*

*Proof.* By definition, we compute

$$\begin{aligned} C_{Th}(\eta, N) &= (\pi_{P_g})_* T^h(K(\bar{\eta}, \bar{N})) = (\pi_{P_g})_*(\tilde{e}(P, \bar{Q}, i_\infty) \cdot \text{Td}_g^{-1}(\bar{Q}) \cdot \text{ch}_g(\pi_P^* \bar{\eta})) \\ &= (\pi_{P_g})_*(\tilde{e}(P, \bar{Q}, i_\infty) \cdot \text{Td}_g^{-1}(\bar{Q})) \cdot \text{ch}_g(\bar{\eta}) = C_{Th}(\mathcal{O}_Y, N) \cdot \text{ch}_g(\eta). \end{aligned}$$

Then this proposition follows from Theorem 7.4  $\square$

The equivariant and non-equivariant homogeneous singular Bott-Chern classes are related by the following proposition.

**Proposition 8.8.** *Let  $S^h$  be the non-equivariant homogeneous singular Bott-Chern classes defined in [5]. Assume that  $\bar{\eta}$  is an equivariant hermitian vector bundle whose restriction to the fixed point submanifold has no non-zero degree part. Then we have*

$$T^h(K(\bar{\eta}, \bar{N})) \cdot \text{Td}_g(\bar{Q}) = S^h(K(\bar{\eta}_g, \bar{N}_g)) \cdot \text{Td}(\bar{Q}_g).$$

*Proof.* We first suppose that  $\bar{\eta}$  is the trivial bundle  $\overline{\mathcal{O}_Y}$  equipped with the trivial  $g$ -structure. By equation [5, (9.8)], we have

$$S^h(K(\overline{\mathcal{O}_{Y_g}}, \bar{N}_g)) = \tilde{e}(P_0, \bar{Q}_g, i_{\infty,0}) \cdot \text{Td}^{-1}(\bar{Q}_g)$$

where  $i_{\infty,0}$  is the zero section embedding from  $Y_g$  to  $P_0 = \mathbb{P}(N_g \oplus \mathcal{O}_{Y_g})$ . Note that by the definition of homogeneity in our paper, the class  $\tilde{e}(P, \bar{Q}, i_\infty)$  is equal to  $\tilde{e}(P_0, \bar{Q}_g, i_{\infty,0})$  in  $\text{Im}(\tilde{\mathcal{U}}^{\text{rk}Q_g-1, \text{rk}Q_g-1}(P_0) \hookrightarrow \tilde{\mathcal{U}}^{\text{rk}Q_g-1, \text{rk}Q_g-1}(P_g))$ . This implies that

$$T^h(K(\overline{\mathcal{O}_Y}, \bar{N})) \cdot \text{Td}_g(\bar{Q}) = S^h(K(\overline{\mathcal{O}_{Y_g}}, \bar{N}_g)) \cdot \text{Td}(\bar{Q}_g).$$

In general case, since the restriction of  $\bar{\eta}$  to  $Y_g$  is supposed to have no non-zero degree part, we have  $\text{ch}_g(\pi_P^* \bar{\eta}) = \pi_{P_g}^* \text{ch}(\bar{\eta}_g)$ . Moreover, the class  $S^h(K(\overline{\mathcal{O}_{Y_g}}, \bar{N}_g)) \cdot \text{Td}(\bar{Q}_g)$  belongs to  $D^{\text{rk}Q_g-1, \text{rk}Q_g-1}(P_0)/(\text{Im}\partial + \text{Im}\bar{\partial})$ , we then can compute

$$T^h(K(\overline{\mathcal{O}_Y}, \bar{N})) \cdot \text{Td}_g(\bar{Q}) \cdot \text{ch}_g(\pi_P^* \bar{\eta}) = S^h(K(\overline{\mathcal{O}_{Y_g}}, \bar{N}_g)) \cdot \text{Td}(\bar{Q}_g) \cdot \pi_{P_0}^* \text{ch}(\bar{\eta}_g).$$

This equality implies that  $T^h(K(\bar{\eta}, \bar{N})) \cdot \text{Td}_g(\bar{Q}) = S^h(K(\bar{\eta}_g, \bar{N}_g)) \cdot \text{Td}(\bar{Q}_g)$  because  $T^h$  and  $S^h$  are both compatible with the projection formula.  $\square$

In general, let  $X$  be a complex manifold and let  $\bar{E}$  be a hermitian holomorphic vector bundle of rank  $r$  on  $X$ . Assume that  $s$  is a holomorphic section of  $E$  which is transversal to the zero section. Denote by  $Y$  the zero locus of  $s$ . In [5, Proposition 9.13], the authors have shown that there is a unique way to attach to each  $(X, \bar{E}, s)$  a class of currents  $\tilde{e}(X, \bar{E}, s) \in \tilde{\mathcal{U}}^{r-1, r-1}(X, N_{Y,0}^\vee)$  which satisfies some axiomatic properties. Such class was also constructed by J.-M. Bismut, H. Gillet and C. Soulé in [2]. We shall use this fact to generalize Proposition 8.8 in the following



way. Assume that all notations above are  $g$ -equivariant, then there is a global equivariant Koszul resolution

$$K(\overline{E}) : 0 \rightarrow \wedge^r \overline{E}^\vee \rightarrow \cdots \rightarrow \overline{E}^\vee \rightarrow \overline{\mathcal{O}}_X \rightarrow i_* \overline{\mathcal{O}}_Y \rightarrow 0.$$

So we get an equivariant hermitian embedded vector bundle  $(i, \overline{N}_{X/Y}, \overline{\mathcal{O}}_Y, K(\overline{E}))$  such that  $\overline{N}_{X/Y}$  is isometric to  $i^* \overline{E}$ . One can carry out the proof of [5, Prop. 9.18] word by word (adding subscript  $g$ ) to prove the following equality

$$T^h(i, \overline{N}_{X/Y}, \overline{\mathcal{O}}_Y, K(\overline{E})) = \tilde{e}(X_g, \overline{E}_g, s_g) \cdot \text{Td}_g^{-1}(\overline{E}).$$

This equality and [5, Prop. 9.18] imply the following result.

**Proposition 8.9.** *Let notations and assumptions be as above, then we have*

$$T^h(i, \overline{N}_{X/Y}, \overline{\mathcal{O}}_Y, K(\overline{E})) \cdot \text{Td}_g(\overline{E}) = S^h(i_g, \overline{N}_{X_g/Y_g}, \overline{\mathcal{O}}_{Y_g}, K(\overline{E}_g)) \cdot \text{Td}(\overline{E}_g).$$

We now recall the construction of the equivariant Bott-Chern singular currents given by J.-M. Bismut in [4]. This construction was realized via some current valued zeta function which involves the supertraces of Quillen's superconnections. We would like to indicate that Bismut's singular current defines a class which agrees with our definition of equivariant singular Bott-Chern class only in some certain situation. Nevertheless, it is easy to use Bismut's results to define a theory of equivariant singular Bott-Chern classes in the sense of Definition 5.5. We shall prove that such a theory is homogeneous.

Let  $i : Y \rightarrow X$  be a closed immersion of equivariant projective manifolds, and let  $\overline{\Xi} = (i, \overline{N}, \overline{\eta}, \overline{\xi})$  be an equivariant hermitian embedded vector bundle. We denote the differential of the complex  $\xi$  by  $v$ . Note that  $\xi$  is acyclic outside  $Y$  and the homology sheaves of its restriction to  $Y$  are locally free. We write  $H_n = \mathcal{H}_n(\xi|_Y)$  and define a  $\mathbb{Z}$ -graded bundle  $H = \bigoplus_n H_n$ . For  $y \in Y$  and  $u \in TX_y$ , we denote by  $\partial_u v(y)$  the derivative of  $v$  at  $y$  in the direction  $u$  in any given holomorphic trivialization of  $\xi$  near  $y$ . Then the map  $\partial_u v(y)$  acts on  $H_y$  as a chain map, and this action only depends on the image  $z$  of  $u$  in  $N_y$ . So we get a chain complex of holomorphic vector bundles  $(H, \partial_z v)$ .

Let  $\pi$  be the projection from the normal bundle  $N$  to  $Y$ , then we have a canonical identification of  $\mathbb{Z}$ -graded chain complexes

$$(\pi^* H, \partial_z v) \cong (\pi^* (\wedge^\bullet N^\vee \otimes \eta), \sqrt{-1} i_z).$$

Moreover, such an identification is an identification of  $g$ -bundles. By finite dimensional Hodge theory, for each  $y \in Y$ , there is a canonical isomorphism

$$H_y \cong \{f \in \xi_y \mid v f = 0, v^* f = 0\}$$

where  $v^*$  is the dual of  $v$  with respect to the metrics on  $\xi$ . This means that  $H$  can be regarded as a smooth  $\mathbb{Z}$ -graded  $g$ -equivariant subbundle of  $\xi$  so that it carries an induced  $g$ -invariant metric. On the other hand, we endow  $\wedge^\bullet N^\vee \otimes \eta$  with the metric induced from  $\overline{N}$  and  $\overline{\eta}$ .

**Definition 8.10.** We say that the metrics on the complex of equivariant hermitian vector bundles  $\bar{\xi}$ . satisfy Bismut assumption (A) if the identification  $(\pi^*H, \partial_z v) \cong (\pi^*(\wedge^\bullet N^\vee \otimes \eta), \sqrt{-1}i_z)$  also identifies the metrics.

**Proposition 8.11.** *There always exist  $g$ -invariant metrics on  $\xi$ . which satisfy Bismut assumption (A) with respect to  $\bar{N}$  and  $\bar{\eta}$ .*

*Proof.* This is [4, Proposition 3.5]. □

Let  $\nabla^\xi$  be the canonical hermitian holomorphic connection on  $\xi$ ., then for  $u > 0$ , we may define a  $g$ -invariant superconnection

$$C_u := \nabla^\xi + \sqrt{u}(v + v^*)$$

on the  $\mathbb{Z}_2$ -graded vector bundle  $\xi$ . Let  $\Phi$  be the map  $\alpha \in \wedge(T_{\mathbb{R}}^*X_g) \rightarrow (2\pi i)^{-\deg \alpha/2} \alpha \in \wedge(T_{\mathbb{R}}^*X_g)$  and denote

$$(\mathrm{Td}_g^{-1})'(\bar{N}) := \frac{\partial}{\partial b} \big|_{b=0} (\mathrm{Td}_g(b \cdot \mathrm{Id} - \frac{\Omega^{\bar{N}}}{2\pi i})^{-1}).$$

**Lemma 8.12.** *Let  $N_H$  be the number operator on the complex  $\xi$ . i.e. it acts on  $\xi_j$  as multiplication by  $j$ , then for  $s \in \mathbb{C}$  and  $0 < \mathrm{Re}(s) < \frac{1}{2}$ , the current valued zeta function*

$$Z_g(\bar{\xi})(s) := \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} [\Phi \mathrm{Tr}_s(N_H g \exp(-C_u^2)) + (\mathrm{Td}_g^{-1})'(\bar{N}) \mathrm{ch}_g(\bar{\eta}) \delta_{Y_g}] du$$

*is well-defined on  $X_g$  and it has a meromorphic continuation to the complex plane which is holomorphic at  $s = 0$ .*

**Definition 8.13.** The equivariant Bott-Chern singular current on  $X_g$  associated to the resolution  $\bar{\xi}$ . is defined as

$$T_g(\bar{\xi}) := \frac{\partial}{\partial s} \big|_{s=0} Z_g(\bar{\xi})(s).$$

**Theorem 8.14.** *The current  $T_g(\bar{\xi})$  is a sum of  $(p, p)$ -currents and it satisfies the differential equation*

$$\mathrm{dd}^c T_g(\bar{\xi}) = i_{g*} \mathrm{ch}_g(\bar{\eta}) \mathrm{Td}_g^{-1}(\bar{N}) - \sum_k (-1)^k \mathrm{ch}_g(\bar{\xi}_k).$$

*Moreover, the wave front set of  $T_g(\bar{\xi})$  is contained in  $N_{g,0}^\vee$ .*

For any equivariant hermitian embedded vector bundle  $\bar{\Xi}_0 = (i, \bar{N}, \bar{\eta}, (\xi, h_0^\xi))$ , we may construct a new embedded bundle  $\bar{\Xi}_1 = (i, \bar{N}, \bar{\eta}, (\xi, h_1^\xi))$  such that the metrics  $h_1^\xi$  satisfies Bismut assumption (A). Then we may attach to  $\bar{\Xi}_0$  an element in  $\tilde{\mathcal{U}}(X_g)$  defined as

$$T^B(\bar{\Xi}_0) = -T_g(\xi, h_1^\xi) + \sum_k (-1)^k \tilde{\mathrm{ch}}_g(\xi_k, h_0^{\xi_k}, h_1^{\xi_k}).$$

**Theorem 8.15.** *The assignment that, to each equivariant hermitian embedded vector bundle  $\overline{\Xi}_0$ , associates the current  $T^B(\overline{\Xi}_0)$ , is a theory of equivariant homogeneous singular Bott-Chern classes.*

*Proof.* We first show that  $T^B(\overline{\Xi}_0)$  is well-defined. Actually, let  $\overline{\Xi}_2 = (i, \overline{N}, \overline{\eta}, (\xi, h_2^\xi))$  be another embedded bundle such that the metrics  $h_2^\xi$  satisfy Bismut assumption (A), then by [14, Theorem 3.14] we have

$$T_g(\overline{\xi}_1) - T_g(\overline{\xi}_2) = - \sum_k (-1)^k \tilde{\text{ch}}_g(\xi_k, h_1^{\xi_k}, h_2^{\xi_k}).$$

Note that we have the equality

$$\tilde{\text{ch}}_g(\xi_k, h_0^{\xi_k}, h_1^{\xi_k}) + \tilde{\text{ch}}_g(\xi_k, h_1^{\xi_k}, h_2^{\xi_k}) + \tilde{\text{ch}}_g(\xi_k, h_2^{\xi_k}, h_0^{\xi_k}) = 0.$$

So we obtain that  $T^B(\overline{\Xi}_0)$  does not depend on the choice of the metrics which satisfy Bismut assumption (A) and hence it is well-defined.

Secondly, the fact that the equivariant singular current  $T^B(\overline{\Xi}_0)$  satisfies the differential equation in Definition 5.5 follows from Theorem 8.14 and the definition of  $\tilde{\text{ch}}_g$ .

The functoriality property for  $T^B(\overline{\Xi}_0)$  follows from the same property for  $T_g$  and for  $\tilde{\text{ch}}_g$ .

For the normalization property, let  $\overline{A}.$  be an equivariantly and orthogonally split exact sequence of equivariant hermitian vector bundles, then using [14, Theorem 3.14] again we have

$$T_g(\overline{\xi} \oplus \overline{A}.) = T_g(\overline{\xi}.) + T_g(\overline{A}.).$$

By [14, Corollary 3.10], if  $\overline{A}.$  is equivariantly and orthogonally split, then  $T_g(\overline{A}.)$  is equal to zero. So by definition we finally get  $T^B(\overline{\Xi} \oplus \overline{A}.) = T^B(\overline{\Xi}.)$ .

At last, by [14, Lemma 3.15], with the hypothesis before Proposition 8.9 we have the following equality

$$T^B(i, \overline{N}_{X/Y}, \overline{\mathcal{O}}_Y, K(\overline{E})) = \tilde{e}(X_g, \overline{E}_g, s_g) \cdot \text{Td}_g^{-1}(\overline{E}) = T^h(i, \overline{N}_{X/Y}, \overline{\mathcal{O}}_Y, K(\overline{E})).$$

Since  $T^B$  and  $T^h$  are both compatible with the projection formula, this equality implies that  $C_{T^B} = C_{T^h}$  and hence  $T^B = T^h$  by Theorem 6.1. So  $T^B$  is homogeneous which completes the whole proof.  $\square$

## 9 Concentration formula

In the last section, we shall prove a concentration formula for equivariant homogeneous singular Bott-Chern class. We call it concentration formula because it can be used to prove a statement which generalizes the concentration theorem in algebraic  $K$ -theory (cf. [17]) to the context of Arakelov geometry. We deal with this in another paper. Before describing the concentration formula, we introduce some basic concepts.

**Definition 9.1.** Let  $X$  be a complex manifold and let  $\bar{\xi}.$  be a bounded complex of hermitian vector bundles on  $X$ . We say  $\bar{\xi}.$  is standard if the homology sheaves of  $\bar{\xi}.$  are all locally free and they are endowed with some hermitian metrics. We shall write a standard complex as  $(\bar{\xi}., h^H)$  to emphasize the choice of the metrics on homology sheaves.

Now let  $X$  be a  $\mu_n$ -equivariant projective manifold, we consider a special closed immersion  $i : X_g \hookrightarrow X$ . For an equivariant hermitian embedded vector bundle  $\bar{\Xi} = (i, \bar{N}, \bar{\eta}, \bar{\xi}.)$ , we always assume that the metrics on  $\bar{\xi}.$  satisfy Bismut assumption (A). In this case, the restriction of  $\bar{\xi}.$  to  $X_g$  is a standard complex according to our discussion in last section, the metrics on homology bundles are induced by the metrics on  $\bar{\xi}.$   $|_{X_g}$ . Note that we can split  $\bar{\xi}.$   $|_{X_g}$  into a series of short exact sequences

$$0 \rightarrow \bar{\text{Im}} \rightarrow \bar{\text{Ker}} \rightarrow \wedge^\bullet \bar{N}^\vee \otimes \bar{\eta} \rightarrow 0$$

and

$$0 \rightarrow \bar{\text{Ker}} \rightarrow \bar{\xi}.$$

Denote the alternating sum of the equivariant secondary characteristic classes of the short exact sequences above by  $\tilde{\text{ch}}_g(\bar{\xi}., h^H)$  such that it satisfies the following differential equation

$$\text{dd}^c \tilde{\text{ch}}_g(\bar{\xi}., h^H) = \text{ch}_g(\bar{\eta}) \text{Td}_g^{-1}(\bar{N}) - \sum_j (-1)^j \text{ch}_g(\bar{\xi}_j).$$

With this observation, we can introduce the following proposition.

**Proposition 9.2.** Let  $\bar{\chi} : 0 \rightarrow \bar{\eta}_n \rightarrow \cdots \rightarrow \bar{\eta}_1 \rightarrow \bar{\eta}_0 \rightarrow 0$  be an exact sequence of equivariant hermitian vector bundles on  $X_g$ , and let  $\bar{\varepsilon} : 0 \rightarrow \bar{\xi}_{n,\cdot} \rightarrow \cdots \rightarrow \bar{\xi}_{1,\cdot} \rightarrow \bar{\xi}_{0,\cdot} \rightarrow 0$  be an exact sequence of resolutions of  $i_* \bar{\chi}$  on  $X$ . As usual we write  $\bar{\varepsilon}_k$  for the exact sequence

$$0 \rightarrow \bar{\xi}_{n,k} \rightarrow \cdots \rightarrow \bar{\xi}_{1,k} \rightarrow \bar{\xi}_{0,k} \rightarrow 0.$$

Then we have the following equality in  $\tilde{A}(X_g)$

$$\sum_{j=0}^n (-1)^j \tilde{\text{ch}}_g(\bar{\xi}_{j,\cdot}, h^H) = \tilde{\text{ch}}_g(\bar{\chi}) \text{Td}_g^{-1}(\bar{N}) - \sum_k (-1)^k \tilde{\text{ch}}_g(\bar{\varepsilon}_k).$$

*Proof.* Note that the fixed point submanifold of  $X \times \mathbb{P}^1$  is exactly  $X_g \times \mathbb{P}^1$ , we know that the construction of the first transgression exact sequence is compatible with restriction to the fixed point submanifold. This means  $\text{tr}_1(\bar{\varepsilon}.)_j |_{X_g \times \mathbb{P}^1}$  is equal to  $\text{tr}_1(\bar{\varepsilon}.)_j |_{X_g}$ . Therefore, one can use the same approach as in the proof of Proposition 5.7 to verify the equality in the statement of this proposition.  $\square$

**Theorem 9.3.** (Concentration formula) Let notations and assumptions be as above. Assume that  $\bar{\Xi} = (i : X_g \rightarrow X, \bar{N}, \bar{\eta}, \bar{\xi}.)$  is an equivariant hermitian embedded vector bundle such that the metrics on  $\bar{\xi}.$  satisfy Bismut assumption (A). Then in  $\tilde{A}(X_g)$ , we have the equality

$$T^h(\bar{\Xi}) = -\tilde{\text{ch}}_g(\bar{\xi}., h^H).$$

Before proving this theorem, we first investigate the problem for a simple case where the hypothesis is the same as before Proposition 8.9. That means there exists an equivariant hermitian vector bundle  $\overline{E}$  on  $X$  which admits a  $g$ -invariant regular section  $s$  such that  $X_g$  is the zero locus of  $s$  and  $i^*\overline{E}$  is isometric to  $\overline{N}_{X/X_g}$ . From this we know that  $\overline{E}_g$  is the zero bundle, so  $S^h(i_g, \overline{N}_{X_g/X_g}, \overline{\mathcal{O}_{X_g}}, K(\overline{E}_g)) = 0$  and hence  $T^h(i, \overline{N}_{X/X_g}, \overline{\mathcal{O}_{X_g}}, K(\overline{E})) = 0$  by Proposition 8.9. On the other hand,  $\tilde{\text{ch}}_g(\wedge^\bullet \overline{E}, h^H)$  is definitely equal to 0 since the metrics  $h^H$  are supposed to be induced from  $\overline{E}$  and these metrics satisfy Bismut assumption (A). Therefore the concentration formula is trivially true for this case.

*Proof.* (of Theorem 9.3) We use the same notations as in the proof of Theorem 6.1, then we have the following expression

$$\begin{aligned} T^h(\Xi) = & -(p_{W_g})_*(U \cdot (\sum_k (-1)^k \text{ch}_g(\text{tr}_1(\tilde{\xi} \cdot)_k) - j_{g*}(\text{ch}_g(p_{X_g}^* \overline{\eta}) \text{Td}_g^{-1}(\overline{N}')))) \\ & - \sum_k (-1)^k (p_{P_g})_*[\tilde{\text{ch}}_g(\tilde{\varepsilon}_k)] + C_{Th}(\eta, N). \end{aligned}$$

Since  $i_g$  is the identity map, we know that the deformation to the normal cone  $W(i_g)$  is equal to  $X_g \times \mathbb{P}^1$ . Moreover  $W(i_g)$  is a disjoint union of some connected components of  $W_g$  and the map  $j_g$  factors through  $W(i_g)$ . We shall write  $W_0$  for  $W(i_g)$  for simplicity and we shall denote by  $W_\perp$  the other components of  $W_g$ . Now we restrict the sum

$$L := \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\tilde{\xi} \cdot)_k) - j_{g*}(\text{ch}_g(p_{X_g}^* \overline{\eta}) \text{Td}_g^{-1}(\overline{N}'))$$

to  $W_\perp$  and  $W_0$ . Over  $W_\perp$  we get  $L|_{W_\perp} = \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\tilde{\xi} \cdot)_k|_{W_\perp})$  which can be rewritten as  $\text{dd}^c \tilde{\text{ch}}_g(\text{tr}_1(\tilde{\xi} \cdot)|_{W_\perp})$ . Similarly, over  $W_0$  we get

$$L|_{W_0} = \sum_k (-1)^k \text{ch}_g(\text{tr}_1(\tilde{\xi} \cdot)_k|_{W_0}) - (\text{ch}_g(p_{X_g}^* \overline{\eta}) \text{Td}_g^{-1}(\overline{N}'))$$

which can be rewritten as  $-\text{dd}^c \tilde{\text{ch}}_g(\text{tr}_1(\tilde{\xi} \cdot)|_{W_0}, h^H)$  since in this case  $\text{tr}_1(\tilde{\xi} \cdot)|_{W_0}$  is clearly a standard complex in the sense of Definition 9.1. Moreover, a totally similar argument to the observation given before this proof shows that  $T^h(K(\overline{\eta}, \overline{N}))$  is equal to 0 so that  $C_{Th}(\eta, N)$  is equal to 0. Furthermore, the exact sequence  $K(\overline{\eta}, \overline{N})|_{W_\perp \cap P}$  is equivariantly and orthogonally split. Therefore, by Remark 2.9, we get

$$-\tilde{\text{ch}}_g(\text{tr}_1(\tilde{\xi} \cdot)|_{W_\perp \cap P}) = \sum_k (-1)^k \tilde{\text{ch}}_g(\tilde{\varepsilon}_k)|_{W_\perp}.$$

This means

$$\begin{aligned} U \cdot L|_{W_\perp} &= U \cdot \text{dd}^c \tilde{\text{ch}}_g(\text{tr}_1(\tilde{\xi} \cdot)|_{W_\perp}) = \text{dd}^c U \cdot \tilde{\text{ch}}_g(\text{tr}_1(\tilde{\xi} \cdot)|_{W_\perp}) \\ &= \tilde{\text{ch}}_g(\text{tr}_1(\tilde{\xi} \cdot)|_{W_\perp \cap P}) = - \sum_k (-1)^k \tilde{\text{ch}}_g(\tilde{\varepsilon}_k)|_{W_\perp}. \end{aligned}$$

Combing these computations above, we may reformulate  $T^h(\Xi)$  as

$$T^h(\Xi) = -(p_{W_0})_*(U \cdot L|_{W_0}) - \sum_k (-1)^k (p_{P_0})_* [\tilde{\text{ch}}_g(\bar{\varepsilon}_k)|_{P_0}].$$

Similar to  $\text{tr}_1(\bar{\xi})|_{W_0}$ ,  $K(\bar{\eta}, \bar{N})|_{P_0}$  is also a standard complex. Since the metrics on the Koszul resolution are supposed to satisfy Bismut assumption (A), we know that  $\tilde{\text{ch}}_g(K(\bar{\eta}, \bar{N})|_{P_0}, h^H)$  is equal to 0. Then by Proposition 9.2, we have that

$$\tilde{\text{ch}}_g(\text{tr}_1(\bar{\xi})|_{W_0 \cap P}, h^H) = \sum_k (-1)^k \tilde{\text{ch}}_g(\bar{\varepsilon}_k|_{P_0}) = \sum_k (-1)^k \tilde{\text{ch}}_g(\bar{\varepsilon}_k)|_{P_0}.$$

Together with the fact that  $\text{tr}_1(\bar{\xi})|_{X_g \times \{0\}}$  is isometric to  $\bar{\xi}|_{X_g}$ , we finally get

$$T^h(\Xi) = -\tilde{\text{ch}}_g(\text{tr}_1(\bar{\xi})|_{W_0}, h^H)|_{X_g \times \{0\}} = -\tilde{\text{ch}}_g(\bar{\xi}, h^H)$$

which completes the proof.  $\square$

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